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Finite propagation speed for solutions of the wave equation on metric graphs

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Abstract

We provide a class of self-adjoint Laplace operators $-\Delta$ on metric graphs with the property that the solutions of the associated wave equation satisfy the finite propagation speed property. The proof uses energy methods, which are adaptations of corresponding methods for smooth manifolds.

Keywords: Metric graphs; Laplace operators; Wave equation; Finite propagation speed

1. Introduction

Nature tells us that energy and information can only be transmitted with finite speed, smaller or equal to the speed of light. The mathematical framework, which allows an analysis and proof of this phenomenon, is the theory of hyperbolic differential equations and in particular of the wave equation

$$\Box\psi = 0$$

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where $\Box = \partial_t^2 - \Delta$ is the d'Alembert operator with $-\Delta$ as the Laplace operator, and $t \in \mathbb{R}$ is a time parameter. The result, which may be obtained, runs under the name *finite propagation speed*. The configuration space and hence the context, within which the wave operator and finite propagation speed can be discussed, may be an arbitrary manifold in which the notions both of a distance between two points and of a Laplace operator makes sense. In more detail, given the Laplacian $-\Delta$ and hence the associated d'Alembert operator, the central quantity entering the construction and discussion of solutions of the wave equation for given Cauchy data (initial conditions) is the wave kernel

$$W(t) = \frac{\sin(\sqrt{-\Delta}t)}{\sqrt{-\Delta}}, \quad t \in \mathbb{R}.$$

Let $W(t)(p,q)$ denote the associated integral kernel. Then finite propagation speed is a general result on hyperbolic equations and the statement that $W(t)(p,q)$ vanishes whenever $|t| <$ distance(p,q). For an extensive text book discussion, see e.g. [4,17,18].

The d'Alembert operator and the associated Klein–Gordon operator $\Box + m^2$ play an important rôle in relativistic quantum theories, see e.g. standard text books on relativistic quantum field theory like [6,15,20]. Free quantum fields of mass $m > 0$ satisfy the Klein–Gordon equation. Thus a quantum version of finite propagation speed is the condition that space-like separated observables commute. Since the fundamental article of Wightman [21], this condition is considered as indispensable for any local relativistic quantum theory [5,7,16]. Thus, the commutator of a Hermitian, free, massive, scalar boson field $\Phi(x,t)$ on Minkowski space $\mathcal{M} = \mathbb{R}^4$ is given by the integral kernel associated to the wave kernel

$$W_m(t) = \frac{\sin(\sqrt{-\Delta + m^2}t)}{\sqrt{-\Delta + m^2}}$$

of the Klein–Gordon operator, that is

$$\bigl[\Phi(\vec{x},t), \Phi(\vec{y},s)\bigr] = i\, W_m(t-s)(\vec{x},\vec{y})\mathbb{I}, \quad (\vec{x},t), (\vec{y},s) \in \mathcal{M}.$$

Two events $(\vec{x},t)$ and $(\vec{y},s)$ are space-like separated if $|\vec{x} - \vec{y}|^2 > (t-s)^2$, in units, where the speed of light equals 1. Thus local commutativity in this context is the property

$$W_m(t-s)(\vec{x},\vec{y}) = 0, \quad \textit{if the events } (\vec{x},t) \textit{ and } (\vec{y},s) \textit{ are space-like separated},$$

which precisely is finite propagation speed.

In this article we prove finite propagation speed when $-\Delta$ is a self-adjoint (s.a.) Laplace operator on a metric graph. Our main result is Theorem 24. Metric graphs are one-dimensional, piecewise linear (p.l.) spaces, and as such they are singular spaces with their vertices viewed as singularities. Previously and to the best of our knowledge finite propagation speed on spaces with singularities has only been proved when the configuration space has conical singularities [3]. As for other applications we mention that in the context of neuronal networks finite propagation speed on axons has been discussed in [1].

We now explain the context and outline in more detail our strategy of proof. As to be expected and as is born out by the proof, finite propagation speed on a metric graph should hold as long as the wave, that is a solution of the wave equation, is located away from the vertices, in other words, as long as one is within a strictly one-dimensional context. Things, however, change, as soon as a vertex is approached, and this gives rise to the first question: How does the wave then split up along the different edges terminating at the vertex? This is where the special properties of the Laplacian $-\Delta$ come into play. Indeed, formally the Laplacian is (minus) the second derivative away from the vertices. To have a s.a. operator, boundary conditions at the vertices have to be specified. In [10,11] a complete characterization of all Laplacians satisfying local boundary conditions was given. By this we mean that each of the boundary conditions involves only the boundary values at one vertex at a time. So these boundary conditions now indeed provide a prescription for how a wave is split up when reaching a vertex.

The second question comes up, when one tries to carry over the standard text book proof of finite propagation speed, which is based on a certain non-negative energy functional, see e.g. [4, 17]. Now this does not work and one has to ask for a remedy. The answer is given by adding terms to the usual energy functional. These terms are localized at the vertices, and they are non-negative provided one chooses the Laplacian $-\Delta$ out of a subset of all (local) s.a. Laplacians. To formulate the corresponding sufficient condition, concepts from Hermitian symplectic geometry have turned out to be useful, see again [10,11]. Thus a s.a. Laplacian, and not only one given in terms of local boundary conditions, can be completely characterized by a maximal isotropic subspace $\mathcal{M}$ in the space of all boundary values, $-\Delta = -\Delta_{\mathcal{M}}$. Hence the sufficient condition on $\mathcal{M}$ is that $\Omega_{\mathcal{M}} \geqslant 0$ holds, where $\Omega_{\mathcal{M}}$ is a quadratic form on the space of boundary values, see Section 2 and in particular relation (2.7). The condition $\Omega_{\mathcal{M}} \geqslant 0$ ensures $-\Delta_{\mathcal{M}} \geqslant 0$ (but not vice versa).

In the usual contexts the self-adjointness of the Laplacian makes the discussion of the existence and the uniqueness of solutions of the wave equation for given L^2 Cauchy data relatively easy. The reason is that this self-adjointness implies nice operator properties of the wave kernel $W(t)$, which are easily obtained with help of the spectral theorem. This is worked out in detail in [2] and our presentation has in a large part been motivated by the discussion given there. Then Sobolev inequalities combined with the ellipticity of the Laplacian form the tools for transforming L^2 properties of the solutions to analytic properties like continuity and differentiability. Our discussion also uses (and needs) Sobolev inequalities in order to control the boundary values since they enter the additional terms in the energy functional just mentioned. The Laplacians we discuss have just this property that Sobolev inequalities can be invoked. As a matter of fact, at the moment we do not know how to deal with the other Laplacians as given and described in [10,11].

Recently one of the authors (R.S.) proved finite propagation speed for an arbitrary s.a. Laplacian on star graphs (possibly having discrete eigenvalues) and on arbitrary metric graphs under two restrictions: (i) $-\Delta \geqslant 0$, and (ii) at least one of the points p or q is on one of the exterior edges [14]. The proof used methods entirely different from the energy estimates usually employed for the proof of finite propagation speed. It is based on properties of the (improper) eigenfunctions of the Laplacians and their analytic properties as functions of the spectral parameter.

The article is organized as follows. In Section 2 we first recall some basic facts about s.a. Laplacians on metric graphs and then we single out those we shall mainly work with. In Section 3 we establish existence and uniqueness of solutions of the wave equation for given Cauchy data. In Section 4 we introduce the local energy functional, which allows us to mimic and modify the standard proof on finite propagation speed on smooth manifolds. Appendix A provides the Sobolev type estimates we need, in particular to control the vertex contributions to the energy functional.

2. Basic structures

In this section we revisit the theory of Laplace operators on a metric graph $\mathcal{G}$. The material presented here is borrowed from the articles [10–12].

A finite graph is a 4-tuple $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial)$, where $\mathcal{V}$ is a finite set of *vertices*, $\mathcal{I}$ is a finite set of *internal edges*, $\mathcal{E}$ is a finite set of *external edges*. For simplicity, from now on when we speak of a graph we will mean a finite graph.

Elements in $\mathcal{I} \cup \mathcal{E}$ are called *edges*. The map ∂ assigns to each internal edge $i \in \mathcal{I}$ an ordered pair of (possibly equal) vertices $\partial(i) := (v_1, v_2)$ and to each external edge $e \in \mathcal{E}$ a single vertex v. The vertices $v_1 =: \partial^-(i)$ and $v_2 =: \partial^+(i)$ are called the *initial* and *final* vertex of the internal edge i, respectively. The vertex $v = \partial(e)$ is the initial vertex of the external edge e. If $\partial(i) = (v, v)$, that is, $\partial^-(i) = \partial^+(i)$ then i is called a *tadpole*. To simplify the discussion, we will exclude tadpoles. Two vertices v and v' are called *adjacent* if there is an internal edge $i \in \mathcal{I}$ such that $v \in \partial(i)$ and $v' \in \partial(i)$. By definition $\operatorname{star}(v) \subseteq \mathcal{V}$ of $v \in \mathcal{V}$ is the set of vertices adjacent to v. A vertex v and the (internal or external) edge $j \in \mathcal{I} \cup \mathcal{E}$ are *incident* if $v \in \partial(j)$.

We do not require the map ∂ to be injective. In particular, any two vertices are allowed to be adjacent to more than one internal edge and two different external edges may be incident with the same vertex. If ∂ is injective and $\partial^-(i) \neq \partial^+(i)$ for all $i \in \mathcal{I}$, the graph $\mathcal{G}$ is called *simple*. The *degree* $\deg(v)$ of the vertex v is defined as

$$\deg(v) = \bigl|\bigl\{e \in \mathcal{E} \bigm| \partial(e) = v\bigr\}\bigr| + \bigl|\bigl\{i \in \mathcal{I} \bigm| \partial^-(i) = v\bigr\}\bigr| + \bigl|\bigl\{i \in \mathcal{I} \bigm| \partial^+(i) = v\bigr\}\bigr|,$$

that is, it is the number of (internal or external) edges incident with the given vertex v. Throughout the whole work we will assume that the graph $\mathcal{G}$ is connected. In particular, this implies that any vertex of the graph $\mathcal{G}$ has nonzero degree, i.e., for any vertex there is at least one edge with which it is incident.

The graph $\mathcal{G}_{\mathrm{int}} = (\mathcal{V}, \mathcal{I}, \emptyset, \partial|_{\mathcal{I}})$ will be called the *interior* of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{I}, \mathcal{E}, \partial)$. It is obtained from $\mathcal{G}$ by eliminating all external edges e. Correspondingly, if $\mathcal{E} \neq \emptyset$, the graph $\mathcal{G}_{\mathrm{ext}} = (\partial\mathcal{V}, \emptyset, \mathcal{E}, \partial|_{\mathcal{E}})$ is called the *exterior* of $\mathcal{G}$. Here $\partial\mathcal{V} \subseteq \mathcal{V}$ is defined to be the set consisting of those vertices v which are of the form $v = \partial(e)$ for some $e \in \mathcal{E}$. We will view both $\mathcal{G}_{\mathrm{int}}$ and $\mathcal{G}_{\mathrm{ext}}$ as subgraphs of $\mathcal{G}$.

We will endow the graph with the following metric structure. Any internal edge $i \in \mathcal{I}$ will be associated with an interval $I_i = [0, a_i]$ with $a_i > 0$ such that the initial vertex of i corresponds to $x = 0$ and the final one to $x = a_i$. Any external edge $e \in \mathcal{E}$ will be associated with a half line $I_e = [0, +\infty)$. We call the number a_i the length of the internal edge i. We make the notational convention that $a_e = \infty$ if $e \in \mathcal{E}$. We will consider the set I_j, $j \in \mathcal{E} \cup \mathcal{I}$ as a subset of $\mathcal{G}$ and write $p \cong (j, x)$ for any point p on I_j with coordinate x. The set of lengths $\{a_i\}_{i \in \mathcal{I}}$, which will also be treated as an element of $\mathbb{R}^{|\mathcal{I}|}$, will be denoted by $\underline{a}$. There is a canonical distance function $d(p, q)$ $(p, q \in \mathcal{G})$ making the graph a metric space. In particular $d(p, q)$ is continuous in both variables. So a graph $\mathcal{G}$ endowed with a metric structure $\underline{a}$ is called a *metric graph*, denoted by $(\mathcal{G}, \underline{a})$. From now on the set $\underline{a}$ of lengths will be fixed and we will simply speak of the metric space $\mathcal{G}$. For given $p \in \mathcal{G}$ and $t > 0$ let $B(p, t)$ denote the closed set of points in $\mathcal{G}$ with distance from p less or equal to t. By definition its boundary $\partial B(p, t)$ is the set of points with distance t from p. Trivially $B(p, t) \subseteq B(p, t')$ for all $t < t'$ (with $B(p, t) \subset B(p, t')$ for all $t < t'$ when $\mathcal{E} \neq \emptyset$) and

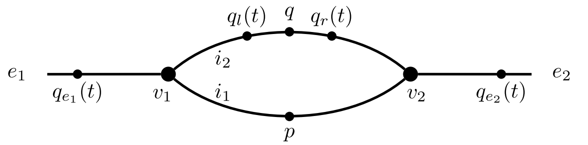

Fig. 1. A point q of coincidence in $\partial B(p,t)$.

$$\lim_{t\uparrow\infty} B(p,t) = \bigcup_{0<t<\infty} B(p,t) = \mathcal{G}.$$

The boundary set $\partial B(p,t)$ deserves special attention. As a function of t the number of elements in $\partial B(p,t)$ is obviously piecewise constant. Here is a partial list of properties. The number of elements in $\partial B(p,t)$ satisfies

$$\bigl|\partial B(p,t)\bigr| = \begin{cases} 2 & \text{for } p \in I_j \setminus \partial I_j,\ 0<t<\operatorname{dist}(p,\partial I_j), \\ \deg(p) & \text{if } p \text{ is a vertex and } t < \operatorname{dist}(p,\operatorname{star}(p)), \\ |\mathcal{E}| & \max_{q\in\mathcal{G}_{\mathrm{int}}} d(p,q) < t. \end{cases}$$

Boundaries at different times have vanishing intersection,

$$\partial B(p,t) \cap \partial B\bigl(p,t'\bigr) = \emptyset, \quad t \neq t'.$$

Fig. 1 provides an example, which serves as a motivation for the following definition.

Definition 1. Given p and t, a point $q \in \partial B(p,t)$ is a *point of coincidence*, if for all $s<t$ sufficiently close to t there are two different points $q_l(s), q_r(s) \in \partial B(p,s)$ such that

$$\lim_{s\uparrow t} q_l(s) = \lim_{s\uparrow t} q_r(s) = q$$

holds. Let $\mathbf{Coin}(p,t) \subseteq \partial B(p,t)$ denote the subset of points of coincidence. Given p, t is *critical* if the set $\mathbf{Coin}(p,t) \cup (\partial B(p,t) \cap \mathcal{V})$ is non-empty. Given p, the set of critical times $t>0$ is denoted by $\mathcal{T}(p)$.

Note that the set $\mathcal{T}(p)$ contains the set of $t \geqslant 0$ at which $|\partial B(p,t)|$ is discontinuous. $\mathcal{T}(p)$ may be strictly larger. As an example consider the case where $\mathcal{G}$ is a star graph with two external edges and vertex v. If t is such that $v \in \partial B(p,t)$, that is $d(v,p)=t$, then $|\partial B(p,t)|$ is continuous at t. More involved examples may easily be constructed. $\mathbf{Coin}(p,t) \cap (\partial B(p,t) \cap \mathcal{V})$ may be non-empty. Also $\mathbf{Coin}(p,t) \subset \mathcal{G}_{\mathrm{int}}$ and if $\mathbf{Coin}(p,t) \cap I_i \neq \emptyset$ for some t and $i \in \mathcal{I}$, then $\mathbf{Coin}(p,t') \cap I_i = \emptyset$ for all $t' \neq t$. Similarly if $v \in \partial B(p,t)$, then $v \notin \partial B(p,t')$ for all $t' \neq t$. From these two observations one easily deduces that $\mathcal{T}(p)$ is a finite set with $|\mathcal{T}(p)| \leqslant |\mathcal{I}| + |\mathcal{V}|$.

Fig. 1 shows the example of a graph with two external edges e_1, e_2, and two internal edges i_1, i_2 of equal length $a = a_{i_1} = a_{i_2}$. There are two vertices v_1 and v_2. Consider a point p on the edge i_1 with coordinate $a/2$. The set $\partial B(p,t)$ consists of 2 points as long as $0<t\leqslant a/2$, of four points when $a/2<t<a$, of three points when $t=a$, and of two points, when $t>a$. Thus $\partial B(p,t) = \{q_l(t), q_r(t), q_{e_1}(t), q_{e_2}(t)\}$ when $a/2<t<a$ and $\partial B(p,t) = \{q_{e_1}(t), q_{e_2}(t)\}$

when $t > a$. The two points $q_l(t)$ and $q_r(t)$ at a distance $a/2 < t < a$ from p, lie on the edge i_2, and collapse to an *antipodal point* q (with coordinate $a/2$) of p, when t increases to a. So $\mathbf{Coin}(p, a) = \{q\}$ holds, while $\mathbf{Coin}(p, t) = \emptyset$ for all $t \neq a$.

In Riemannian geometry there is an analogue to the notion of a point of coincidence. It arises in the context of geodesics and is given by the notion of a conjugate point. Thus a time t, for which $\mathbf{Coin}(p, t) \neq \emptyset$ while $\mathbf{Coin}(p, t') = \emptyset$ for all $t' < t$, is the analogue of the injectivity radius, that is the radius at which the exponential map ceases to be injective.

There is a canonical Lebesgue measure $\mathcal{G}$, so that the notion of $L^p(\mathcal{G})$ spaces of measurable functions on $\mathcal{G}$ makes sense. More generally, we will consider the spaces $L^p(\mathcal{F})$ where $\mathcal{F}$ is any measurable subset of $\mathcal{G}$ and use the notation

$$\int_{\mathcal{F}} \psi(p)\, dp$$

to describe the integral of an element $\psi \in L^1(\mathcal{F})$ and the notation

$$\langle \varphi, \psi \rangle_{\mathcal{F}} = \int_{\mathcal{F}} \overline{\varphi(p)} \psi(p)\, dp$$

to describe the scalar product of two elements φ, ψ in the Hilbert space $L^2(\mathcal{F})$. Also we write $\|\psi\|^2_{\mathcal{F}} = \langle \psi, \psi \rangle_{\mathcal{F}}$. Whenever the context is clear we will simply write $\|\psi\|^2$ and $\langle \varphi, \psi \rangle$ for $\|\psi\|^2_{\mathcal{G}}$ and $\langle \varphi, \psi \rangle_{\mathcal{G}}$ respectively. There is an alternative way to obtain $L^2(\mathcal{G})$, which is useful for the discussion of Laplace operators. The central idea is to consider for any measurable function ψ on $\mathcal{G}$ its restriction ψ_i to the edge I_i, $i \in \mathcal{E} \cup \mathcal{I}$.

So consider the Hilbert space

$$\mathcal{H} \equiv \mathcal{H}(\mathcal{E}, \mathcal{I}, \underline{a}) = \mathcal{H}_{\mathcal{E}} \oplus \mathcal{H}_{\mathcal{I}}, \quad \mathcal{H}_{\mathcal{E}} = \bigoplus_{e \in \mathcal{E}} \mathcal{H}_e,\ \mathcal{H}_{\mathcal{I}} = \bigoplus_{i \in \mathcal{I}} \mathcal{H}_i,$$

where $\mathcal{H}_e = L^2([0, \infty))$ for all $e \in \mathcal{E}$ and $\mathcal{H}_i = L^2([0, a_i])$ for all $i \in \mathcal{I}$. Then $L^2(\mathcal{G}) \cong \mathcal{H}$ holds and from now on we shall interchangeably work with both notations. Moreover, to keep our notation simple, we shall identify I_i with the interval $[0, +\infty)$ if $i \in \mathcal{E}$ and with $[0, a_i]$ if $i \in \mathcal{I}$, unless there is danger of confusion. Of course the spaces $L^p(\mathcal{G})$ have a similar alternative description.

By $\mathcal{D}_i$ with $i \in \mathcal{E} \cup \mathcal{I}$ we denote the set of all $\psi_i \in \mathcal{H}_i$ such that ψ_i and its derivative ψ_i' are absolutely continuous, and its second derivative ψ_i'' is square integrable. Let $\mathcal{D}_i^0$ denote the set of those elements $\psi_i \in \mathcal{D}_i$ which satisfy

$$\begin{array}{l} \psi_i(0) = 0 \\ \psi_i'(0) = 0 \end{array} \quad \text{for } i \in \mathcal{E}, \quad \text{and} \quad \begin{array}{l} \psi_i(0) = \psi_i(a_i) = 0 \\ \psi_i'(0) = \psi_i'(a_i) = 0 \end{array} \quad \text{for } i \in \mathcal{I}.$$

Let Δ^0 be the differential operator

$$\left(\Delta^0 \psi\right)_i(x) = \psi_i''(x), \quad x \in I_i,\ i \in \mathcal{I} \cup \mathcal{E}, \tag{2.1}$$

with domain

$$\mathcal{D}^0 = \bigoplus_{i\in\mathcal{E}\cup\mathcal{I}} \mathcal{D}_i^0 \subset \mathcal{H}.$$

It is straightforward to verify that Δ^0 is a closed symmetric operator with deficiency indices equal to $|\mathcal{E}| + 2|\mathcal{I}|$.

We introduce an auxiliary finite-dimensional Hilbert space

$$\mathcal{K} \equiv \mathcal{K}(\mathcal{E},\mathcal{I}) = \mathcal{K}_{\mathcal{E}} \oplus \mathcal{K}_{\mathcal{I}}^{(-)} \oplus \mathcal{K}_{\mathcal{I}}^{(+)} \tag{2.2}$$

with $\mathcal{K}_{\mathcal{E}} \cong \mathbb{C}^{|\mathcal{E}|}$ and $\mathcal{K}_{\mathcal{I}}^{(\pm)} \cong \mathbb{C}^{|\mathcal{I}|}$. Let ${}^d\mathcal{K}$ denote the "double" of $\mathcal{K}$, that is, ${}^d\mathcal{K} = \mathcal{K} \oplus \mathcal{K}$.

For any

$$\psi \in \mathcal{D} := \bigoplus_{i\in\mathcal{E}\cup\mathcal{I}} \mathcal{D}_i$$

we set

$$[\psi] := \underline{\psi} \oplus \underline{\psi}' \in {}^d\mathcal{K},$$

with the boundary values $\underline{\psi}$ and $\underline{\psi}'$ defined by

$$\begin{aligned}
\underline{\psi} &= \big((\psi_e,\ e\in\mathcal{E}), \big(\psi_i(0),\ i\in\mathcal{I}\big), \big(\psi_i(a_i),\ i\in\mathcal{I}\big)\big)^t,\\
\underline{\psi}' &= \big(\big(\psi_e',\ e\in\mathcal{E}\big), \big(\psi_i'(0),\ i\in\mathcal{I}\big), \big(-\psi_i'(a_i),\ i\in\mathcal{I}\big)\big)^t.
\end{aligned} \tag{2.3}$$

Here the superscript t denotes transposition. Let J be the canonical symplectic matrix on ${}^d\mathcal{K}$,

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

with $\mathbb{I}$ being the identity operator on $\mathcal{K}$. Consider the non-degenerate Hermitian symplectic form

$$\omega\big([\varphi], [\psi]\big) := \big\langle [\varphi], J[\psi]\big\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in ${}^d\mathcal{K} \cong \mathbb{C}^{2(|\mathcal{E}|+2|\mathcal{I}|)}$.

A linear subspace $\mathcal{M}$ of ${}^d\mathcal{K}$ is called *isotropic* if the form ω vanishes on $\mathcal{M}$ identically. An isotropic subspace is called *maximal* if it is not a proper subspace of a larger isotropic subspace. Every maximal isotropic subspace has complex dimension equal to $|\mathcal{E}| + 2|\mathcal{I}|$.

Let A and B be linear maps of $\mathcal{K}$ onto itself. By (A, B) we denote the linear map from ${}^d\mathcal{K} = \mathcal{K} \oplus \mathcal{K}$ to $\mathcal{K}$ defined by the relation

$$(A, B)(\chi_1 \oplus \chi_2) := A\chi_1 + B\chi_2,$$

where $\chi_1, \chi_2 \in \mathcal{K}$. Set

$$\mathcal{M}(A, B) := \operatorname{Ker}(A, B).$$

Theorem 2. *A subspace $\mathcal{M} \subset {}^d\mathcal{K}$ is maximal isotropic if and only if there exist linear maps $A, B : \mathcal{K} \to \mathcal{K}$ such that $\mathcal{M} = \mathcal{M}(A, B)$ and*

$$\begin{aligned} &\text{(i) the map } (A, B) : {}^d\mathcal{K} \to \mathcal{K} \text{ has maximal rank equal to } |\mathcal{E}| + 2|\mathcal{I}|,\\ &\text{(ii) } AB^\dagger \text{ is self-adjoint, } AB^\dagger = BA^\dagger. \end{aligned} \tag{2.4}$$

A proof is given in [11]. The boundary conditions (A, B) and (A', B') satisfying (2.4) are called equivalent if the corresponding maximal isotropic subspaces coincide, that is, $\mathcal{M}(A, B) = \mathcal{M}(A', B')$, and this in turn holds if and only if there is an invertible C such that $A' = CA$, $B' = CB$ is valid.

There is a one-to-one correspondence between all self-adjoint extensions of Δ^0 and maximal isotropic subspaces $\mathcal{M}$ of ${}^d\mathcal{K}$ (see [10,11]). In explicit terms, any self-adjoint extension of Δ^0 is the differential operator defined by (2.1) with domain

$$\mathrm{Dom}(\Delta) = \bigl\{\psi \in \mathcal{D} \bigm| [\psi] \in \mathcal{M}\bigr\}, \tag{2.5}$$

where $\mathcal{M}$ is a maximal isotropic subspace of ${}^d\mathcal{K}$. Conversely, any maximal isotropic subspace $\mathcal{M}$ of ${}^d\mathcal{K}$ defines through (2.5) a self-adjoint operator $\Delta_{\mathcal{M}}$. In the sequel we will call the operator $\Delta_{\mathcal{M}}$ a Laplace operator on the metric graph $\mathcal{G}$. Thus we have $\Delta_{\mathcal{M}}\psi = \psi''$ and in particular

$$\bigl\|\psi''\bigr\| = \|\Delta_{\mathcal{M}}\psi\| \quad \text{for } \psi \in \mathcal{D}(\Delta_{\mathcal{M}}). \tag{2.6}$$

From the discussion above it follows immediately that any self-adjoint Laplace operator on $\mathcal{H}$ equals $\Delta_{\mathcal{M}}$ for some maximal isotropic subspace $\mathcal{M}$. Moreover, $\Delta_{\mathcal{M}} = \Delta_{\mathcal{M}'}$ if and only if $\mathcal{M} = \mathcal{M}'$. For short we will henceforth call $\mathcal{M}$ a boundary condition. The role of the Hermitian symplectic form ω is clarified by the following observation. Consider the Hermitian symplectic form $\widehat{\omega}$ on $\mathcal{D}$

$$\widehat{\omega}(\varphi, \psi) = \bigl(\Delta^0\varphi, \psi\bigr) - \bigl(\varphi, \Delta^0\psi\bigr).$$

Then by Green's theorem

$$\widehat{\omega}(\varphi, \psi) = \omega\bigl([\varphi], [\psi]\bigr)$$

holds, such that $\widehat{\omega}$ vanishes on $\mathrm{Dom}(\Delta_{\mathcal{M}})$.

All operators $-\Delta_{\mathcal{M}}$ are finite rank perturbations of each other and in particular bounded from below. So any $-\Delta_{\mathcal{M}}$ has absolutely continuous spectrum, equal to the positive real axis and with multiplicity $|\mathcal{E}|$. By definition the boundary condition $\mathcal{M}$ is *real* if there are real matrices A and B such that $\mathcal{M} = \mathcal{M}(A, B)$. For real $\mathcal{M}$, the Laplacian $-\Delta_{\mathcal{M}}$ is also real in the sense that for all $\psi \in \mathrm{Dom}(-\Delta_{\mathcal{M}})$ also $\overline{\psi} \in \mathrm{Dom}(-\Delta_{\mathcal{M}})$ and $\overline{-\Delta_{\mathcal{M}}\psi} = -\Delta_{\mathcal{M}}\overline{\psi}$. For more details, see [10,11].

For given $\mathcal{M} = \mathcal{M}(A, B)$ the orthogonal projector $P_{\mathcal{M}}$ in ${}^d\mathcal{K}$ onto $\mathcal{M}$ is given as

$$\begin{aligned} P_{\mathcal{M}} &= \begin{pmatrix} -B^\dagger \\ A^\dagger \end{pmatrix} \bigl(AA^\dagger + BB^\dagger\bigr)^{-1}(-B, A) \\ &= \begin{pmatrix} B^\dagger(AA^\dagger + BB^\dagger)^{-1}B & -B^\dagger(AA^\dagger + BB^\dagger)^{-1}A \\ -A^\dagger(AA^\dagger + BB^\dagger)^{-1}B & A^\dagger(AA^\dagger + BB^\dagger)^{-1}A \end{pmatrix}, \end{aligned}$$

where the block matrix notation is used with respect to the orthogonal decomposition ${}^d\mathcal{K} = \mathcal{K} \oplus \mathcal{K}$. With the same decomposition define

$$\Omega = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix}$$

and set $\Omega_{\mathcal{M}} = P_{\mathcal{M}} \Omega P_{\mathcal{M}}$, giving

$$\Omega_{\mathcal{M}} = -\begin{pmatrix} -B^\dagger \\ A^\dagger \end{pmatrix} \bigl(AA^\dagger + BB^\dagger\bigr)^{-1} AB^\dagger \bigl(AA^\dagger + BB^\dagger\bigr)^{-1} (-B, A), \tag{2.7}$$

a Hermitian $2(|\mathcal{E}| + 2|\mathcal{I}|) \times 2(|\mathcal{E}| + 2|\mathcal{I}|)$ matrix. Observe that Ω is *half* of the canonical symplectic matrix J in the sense that $J = \Omega - \Omega^\dagger$ holds. Now $\Omega_{\mathcal{M}} = \Omega_{\mathcal{M}}^\dagger = P_{\mathcal{M}} \Omega^\dagger P_{\mathcal{M}}$ and hence $P_{\mathcal{M}} J P_{\mathcal{M}} = 0$, another way of stating that the space $\mathcal{M}$ is isotropic.

We quote the following result from [12].

Proposition 3. *For any maximal isotropic subspace $\mathcal{M} \subset {}^d\mathcal{K}$ the identity*

$$\langle \varphi, -\Delta_{\mathcal{M}} \psi \rangle_{\mathcal{G}} = \bigl\langle \varphi', \psi' \bigr\rangle_{\mathcal{G}} + \bigl\langle [\varphi], \Omega_{\mathcal{M}} [\psi] \bigr\rangle_{{}^d\mathcal{K}} \tag{2.8}$$

holds for all $\varphi, \psi \in \operatorname{Dom}(-\Delta_{\mathcal{M}})$.

Observe that by the identity (2.8) $\|\psi'\|_{\mathcal{G}}$ is finite for any $\psi \in \operatorname{Dom}(-\Delta_{\mathcal{M}})$. Indeed, the boundary values $[\psi]$ and $[\psi]$ exist so that $\langle [\psi], \Omega_{\mathcal{M}} [\psi] \rangle_{{}^d\mathcal{K}}$ is well-defined and finite because $\operatorname{Dom}(-\Delta_{\mathcal{M}}) \subset \mathcal{D}$. This proposition immediately gives the first part of

Corollary 4. *If the boundary condition $\mathcal{M}$ is such that $\Omega_{\mathcal{M}} \geqslant 0$, then also $-\Delta_{\mathcal{M}} \geqslant 0$. If $\Omega_{\mathcal{M}} > 0$, then 0 is not an eigenvalue of $-\Delta_{\mathcal{M}}$.*

Proof. To prove the second part, assume there is $\psi \in \operatorname{Dom}(-\Delta_{\mathcal{M}})$ with $-\Delta_{\mathcal{M}} \psi = 0$. (2.8) gives $\psi' = 0$. So ψ has to be constant on each edge. But (2.8) also implies $[\psi] = 0$ which is only possible, if $\psi = 0$. □

The converse does not hold, that is $-\Delta_{\mathcal{M}} \geqslant 0$ does not imply $\Omega_{\mathcal{M}} \geqslant 0$, as Example 3.8 in [9] shows.

The next corollary is a trivial consequence of (2.7) and (2.8) in combination with Theorem 2.

Corollary 5. *If the boundary condition $\mathcal{M}$ is such that $\Omega_{\mathcal{M}} \geqslant 0$ and hence $-\Delta_{\mathcal{M}} \geqslant 0$ is valid, then*

$$\bigl\|\psi'\bigr\|_{\mathcal{G}} \leqslant \bigl\|\sqrt{-\Delta_{\mathcal{M}}}\,\psi\bigr\|_{\mathcal{G}} \tag{2.9}$$

holds.

If $\Omega_{\mathcal{M}} > 0$ and if the boundary value $[\psi]$ is non-vanishing, then the inequality is strict. $\Omega_{\mathcal{M}} = 0$ if and only if $\mathcal{M} = \mathcal{M}(A, B)$ is such that $AB^\dagger = 0$ holds and then (2.9) is actually an equality for all $\psi \in \operatorname{Dom}(\sqrt{-\Delta_{\mathcal{M}}})$.

The inequality (2.9) for the norm of the first derivative compares with the identity (2.6) for the second derivative. Characterizations of maximal isotropic subspaces $\mathcal{M}(A, B)$ satisfying $AB^{\dagger} = 0$ are given in [8, Proposition 2.4] and [11, Remark 3.9]. There also examples are provided.

With respect to the decomposition (2.2) any vector χ in $\mathcal{K}$ can be represented as

$$\chi = \big((\chi_e,\ e \in \mathcal{E}), \big(\chi_i^-,\ i \in \mathcal{I}\big), \big(\chi_i^+,\ i \in \mathcal{I}\big)\big)^t. \tag{2.10}$$

Consider the orthogonal decomposition

$$\mathcal{K} = \bigoplus_{v \in \mathcal{V}} \mathcal{L}_v$$

with $\mathcal{L}_v$ being the linear subspace of dimension $\deg(v)$ spanned by those elements χ in $\mathcal{K}$ of the form (2.10) which satisfy

$$\begin{aligned}
&\chi_e = 0, \quad \text{if } e \in \mathcal{E} \text{ is not incident with } v,\\
&\chi_i^- = 0, \quad \text{if } v \text{ is not an initial vertex of } i \in \mathcal{I},\\
&\chi_i^+ = 0, \quad \text{if } v \text{ is not a final vertex of } i \in \mathcal{I}.
\end{aligned}$$

Set ${}^d\mathcal{L}_v := \mathcal{L}_v \oplus \mathcal{L}_v \cong \mathbb{C}^{2\deg(v)}$. Obviously each ${}^d\mathcal{L}_v$ inherits in a canonical way a symplectic structure from ${}^d\mathcal{K}$ such that the orthogonal and symplectic decomposition

$$\bigoplus_{v \in \mathcal{V}} {}^d\mathcal{L}_v = {}^d\mathcal{K}$$

holds. If the boundary condition $\mathcal{M} = \mathcal{M}(A, B)$ is *local* (see [11] for more details), then A and B have a decomposition

$$A = \bigoplus_{v \in \mathcal{V}} A_v, \qquad B = \bigoplus_{v \in \mathcal{V}} B_v,$$

that is A_v and B_v are linear transformations on $\mathcal{L}_v$, such that (A_v, B_v) is a linear transformation in ${}^d\mathcal{L}_v$. So correspondingly there is a decomposition

$$\mathcal{M} = \bigoplus_{v \in \mathcal{V}} \mathcal{M}_v$$

with $\mathcal{M}_v = \mathcal{M}(A_v, B_v) = \ker(A_v, B_v)$. Let Q_v denote the orthogonal projection in $\mathcal{K}$ onto $\mathcal{L}_v$ and ${}^dQ_v := Q_v \oplus Q_v$ its double, that is the orthogonal projection in ${}^d\mathcal{K}$ onto ${}^d\mathcal{L}_v$. Then we have

Lemma 6. *The relation*

$${}^dQ_v P_{\mathcal{M}} = P_{\mathcal{M}} {}^dQ_v \tag{2.11}$$

holds and equals the orthogonal projection P_v in ${}^d\mathcal{K}$ onto $\mathcal{M}_v$.

Note that (2.11) implies that ${}^d Q_v\, P_{\mathcal{M}}$ is an orthogonal projector. Also $P_v P_{v'} = P_{v'} P_v$, so in particular the P_v's commute pairwise.

Proof. Let $\mathcal{P}_{\mathcal{M}_v}$ denote the orthogonal projection in ${}^d\mathcal{L}_v$ onto $\mathcal{M}_v$, that is $\mathcal{P}_{\mathcal{M}_v}$ is obtained in a similar way from (A_v, B_v) as is $\mathcal{P}_{\mathcal{M}}$ from the pair (A, B) with ${}^d\mathcal{K}$ being replaced by ${}^d\mathcal{L}_v$. Denote by P_v the orthogonal projection in ${}^d\mathcal{K}$ onto $\mathcal{M}_v$. Then $P_{\mathcal{M}_v}$ is the restriction of $\mathcal{P}_{\mathcal{M}}$ to ${}^d\mathcal{L}_v$, $P_{\mathcal{M}_v} = \mathcal{P}_{\mathcal{M}}|_{{}^d\mathcal{L}_v}$. Similarly we view ${}^d Q_v$ as a map from ${}^d\mathcal{K}$ onto ${}^d\mathcal{L}_v$ and its restriction ${}^d Q_v|_{\mathcal{M}}$ to $\mathcal{M}$ as map from $\mathcal{M}$ onto $\mathcal{M}_v$. Then we have a commutative diagram

$$\begin{tikzcd}
{}^d\mathcal{K} \arrow[r, "\mathcal{P}_{\mathcal{M}}"] \arrow[d, "{}^d Q_v"'] & \mathcal{M} \arrow[d, "{}^d Q_v|_{\mathcal{M}}"] \\
{}^d\mathcal{L}_v \arrow[r, "\mathcal{P}_{\mathcal{M}_v} = \mathcal{P}_{\mathcal{M}}|_{{}^d\mathcal{L}_v}"] & \mathcal{M}_v
\end{tikzcd}$$

from which (2.11) and the equality $P_v = {}^d Q_v P_{\mathcal{M}}$ follow. □

Correspondingly we obtain

$$\Omega_{\mathcal{M}} = \bigoplus_{v\in\mathcal{V}} \Omega_v$$

with $\Omega_v = P_v\Omega = \Omega P_v = P_v\Omega P_v$. Since $\mathcal{M}_v$ is a subspace of $\mathcal{M}$ also

$$P_v\mathcal{P}_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}} P_v = {}^d Q_v \mathcal{P}_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}{}^d Q_v = P_v \tag{2.12}$$

holds. For any subset $\mathcal{V}'$ of $\mathcal{V}$ we introduce the orthogonal projection $P_{\mathcal{V}'} = \bigoplus_{v\in\mathcal{V}'} P_v$. These $P_{\mathcal{V}'}$'s commute pairwise. Finally set

$$\Omega_{\mathcal{M},\mathcal{V}'} = P_{\mathcal{V}'}\Omega_{\mathcal{M}} = P_{\mathcal{V}'}\Omega_{\mathcal{M}}P_{\mathcal{V}'} = \Omega_{\mathcal{M}}P_{\mathcal{V}'} \tag{2.13}$$

such that $\Omega_{\mathcal{M},\mathcal{V}} = \Omega_{\mathcal{M}}$ holds. More generally

$$\Omega_{\mathcal{M},\mathcal{V}''} = P_{\mathcal{V}''}\Omega_{\mathcal{M},\mathcal{V}'}P_{\mathcal{V}''}$$

is valid for any pair $\mathcal{V}'' \subseteq \mathcal{V}'$. The following lemma is trivial

Lemma 7. *$\Omega_{\mathcal{M}} \geqslant 0$ is valid if and only if $\Omega_{\mathcal{M},v} \geqslant 0$ for all $v \in \mathcal{V}$. Similarly $\Omega_{\mathcal{M}} > 0$ holds if and only if $\Omega_{\mathcal{M},v} > 0$ for all $v \in \mathcal{V}$. If $\Omega_{\mathcal{M}} \geqslant 0$ then $0 \leqslant \Omega_{\mathcal{M},\mathcal{V}''} \leqslant \Omega_{\mathcal{M},\mathcal{V}'}$ for all $\mathcal{V}'' \subseteq \mathcal{V}'$.*

3. Existence and uniqueness of solutions of the wave equation

Throughout this section we fix a maximal isotropic subspace $\mathcal{M}$ of ${}^d\mathcal{K}$. We introduce the wave kernel

$$W(t) = W_{\mathcal{M}}(t) = \frac{\sin(\sqrt{-\Delta_{\mathcal{M}}}\,t)}{\sqrt{-\Delta_{\mathcal{M}}}}, \quad t \in \mathbb{R}$$

which is defined through the spectral representation of the self-adjoint operator $-\Delta_{\mathcal{M}}$ and operator calculus. Note that at the moment we do not (need to) assume $-\Delta_{\mathcal{M}}$ to be non-negative. $W(t)$ is bounded and self-adjoint for all $t \in \mathbb{R}$ with $W(0) = 0$. Set

$$\rho_{\mathcal{M}}(t) = \cosh(t\sqrt{-\varepsilon_{\mathcal{M}}})$$

with

$$\varepsilon_{\mathcal{M}} = \min\bigl(\inf \operatorname{spec}(-\Delta_{\mathcal{M}}), 0\bigr).$$

$\rho_{\mathcal{M}}(t) = 1$ for all $t \in \mathbb{R}$, if $-\Delta_{\mathcal{M}}$ is non-negative. $W(t)$ is a bounded operator and norm continuous in t

$$\bigl\|W(t)\bigr\| \leqslant |t|\rho_{\mathcal{M}}(t), \qquad \bigl\|W(t) - W\bigl(t'\bigr)\bigr\| \leqslant \bigl|t - t'\bigr| \max\bigl(\rho_{\mathcal{M}}(t), \rho_{\mathcal{M}}\bigl(t'\bigr)\bigr). \tag{3.1}$$

We will also consider the time derivatives of the wave kernel $W(t)$:

$$\begin{aligned} \partial_t W(t) &= \cos(\sqrt{-\Delta_{\mathcal{M}}}t), \\ \partial_t^2 W(t) &= -\sqrt{-\Delta_{\mathcal{M}}} \sin(\sqrt{-\Delta_{\mathcal{M}}}t) = \Delta_{\mathcal{M}} W(t), \end{aligned} \tag{3.2}$$

where the derivatives are taken in the strong operator topology. $\partial_t W(t)$ is a bounded self-adjoint operator for all t with operator norm bound

$$\bigl\|\partial_t W(t)\bigr\| \leqslant \rho_{\mathcal{M}}(t), \quad t \in \mathbb{R}. \tag{3.3}$$

The estimates (3.1) and (3.3) follow from the spectral theorem and the trivial bounds

$$\sup_{x\in\mathbb{R}} \left|\frac{\sin x}{x}\right| \leqslant 1, \qquad \sup_{0\leqslant x\leqslant y} \frac{\sinh x}{x} \leqslant \cosh y, \quad y \geqslant 0.$$

It will be convenient to introduce the following notation. Choose $m^2 \geqslant 0$ such that $-\Delta_{\mathcal{M}} + m^2$ is non-negative. In particular, if $-\Delta_{\mathcal{M}} \geqslant 0$, we set $m^2 = 0$.

For any $\alpha \geqslant 0$ the domain $\operatorname{Dom}((-\Delta_{\mathcal{M}} + m^2)^{\alpha/2})$ may be equipped with the inner product

$$\langle\langle \phi, \psi \rangle\rangle = \langle \phi, \psi \rangle_{\mathcal{G}} + \bigl\langle \bigl(-\Delta_{\mathcal{M}} + m^2\bigr)^{\alpha/2} \phi, \bigl(-\Delta_{\mathcal{M}} + m^2\bigr)^{\alpha/2} \psi \bigr\rangle_{\mathcal{G}},$$

turning it into a Hilbert space, which we denote by $H^{\alpha}_{\mathcal{M},m^2} \subset L^2(\mathcal{G})$. The relation $H^{\alpha}_{\mathcal{M},m^2} \subset H^{\alpha'}_{\mathcal{M},m^2}$ whenever $\alpha' \leqslant \alpha$ is obvious. By construction the semi-norm on $H^{\alpha}_{\mathcal{M},m^2}$

$$\|\psi\|_{\mathcal{M},\alpha} = \bigl\|\bigl(-\Delta_{\mathcal{M}} + m^2\bigr)^{\alpha/2} \psi\bigr\|_{\mathcal{G}}$$

satisfies $\|\psi\|^2_{\mathcal{M},\alpha} \leqslant \langle\langle \psi, \psi \rangle\rangle$ and is hence a continuous map from $H^{\alpha}_{\mathcal{M},m^2}$ onto the non-negative numbers. It is a norm if and only if 0 is not an eigenvalue of $-\Delta_{\mathcal{M}} + m^2$. For varying α these Sobolev (semi-)norms will constitute the basic tools when we estimate solutions of the wave equation in terms of the initial data and to which we turn now.

For $k \in \mathbb{N}$, an iteration of (3.2) yields

$$\partial_t^{2k+1} W(t) = \Delta_{\mathcal{M}}^k \cos(\sqrt{-\Delta_{\mathcal{M}}} t) = \Delta_{\mathcal{M}}^k \partial_t W(t),$$
$$\partial_t^{2k} W(t) = (-1)^k (\sqrt{-\Delta_{\mathcal{M}}})^{2k-1} \sin(\sqrt{-\Delta_{\mathcal{M}}} t) = \Delta_{\mathcal{M}}^k W(t). \tag{3.4}$$

For $n \in \mathbb{Z}$ set $n_+ = \max(n, 0)$. An easy application of the spectral theorem gives the following

Lemma 8.

(a) *For all $n \in \mathbb{N}_0$, $t \in \mathbb{R}$, $\partial_t^n W(t)$ is a self-adjoint operator with domain*

$$\operatorname{Dom}\big(\partial_t^n W(t)\big) = H_{\mathcal{M},m^2}^{(n-1)_+},$$

commuting with $\Delta_{\mathcal{M}}$, and mapping $H_{\mathcal{M},m^2}^{(l+n-1)_+}$ into $H_{\mathcal{M},m^2}^l$, $l \in \mathbb{N}_0$. Moreover, $\partial_t^n W(t)$ is strongly continuous in t as an operator on $H_{\mathcal{M},m^2}^{(n-1)_+}$.

(b) *For all $n \in \mathbb{N}_0$, $t \in \mathbb{R}$, the relation*

$$\Box_{\mathcal{M}} \partial_t^n W(t) = 0$$

holds on $H_{\mathcal{M},m^2}^{n+1}$, where $\Box_{\mathcal{M}}$ is the d'Alembert operator *associated with $\Delta_{\mathcal{M}}$:*

$$\Box_{\mathcal{M}} = \partial_t^2 - \Delta_{\mathcal{M}}.$$

For any *Cauchy data* $(\psi_0, \dot{\psi}_0)$ in $L^2(\mathcal{G}) \times L^2(\mathcal{G})$ set

$$\psi(t) = \partial_t W(t)\psi_0 + W(t)\dot{\psi}_0, \quad t \in \mathbb{R}, \tag{3.5}$$

which is a well-defined element in $L^2(\mathcal{G})$ for all t, strongly continuous in t. For what follows it will be convenient to introduce the notation

$$H_{\mathcal{M},m^2}^{\alpha,\beta} = H_{\mathcal{M},m^2}^{\alpha} \oplus H_{\mathcal{M},m^2}^{\beta}, \quad \alpha, \beta \geqslant 0.$$

If the Cauchy data $(\psi_0, \dot{\psi}_0)$ belong to $H_{\mathcal{M},m^2}^{2,1}$, then we obtain from Lemma 8 that for all t, $\psi(t)$ is twice strongly differentiable in t, that it belongs to $H_{\mathcal{M},m^2}^2$, and that it is a solution of the initial value problem of the wave equation

$$\Box_{\mathcal{M}} \psi(t) = 0,$$
$$\psi(t=0) = \psi_0,$$
$$\partial_t \psi(t=0) = \dot{\psi}_0. \tag{3.6}$$

$\psi(t) \in H_{\mathcal{M},m^2}^2$ implies that $\psi(t)$ and $\psi(t)'$ are absolutely continuous on every open edge for all t. Also the boundary values $[\psi(t)]$ at the vertices of $\mathcal{G}$ may be taken. If $(\psi_0, \dot{\psi}_0) \in H_{\mathcal{M},m^2}^{3,2}$, then an analogous statement is true for $\partial_t\psi(t)$, and both $\psi(t)$ and $\partial_t\psi(t)$ have for all t second order spatial derivatives on the open edges, which define elements in $L^2(\mathcal{G})$. Therefore in this case the set of boundary values $[\partial_t\psi(t)]$ exists, too. If both the boundary condition $\mathcal{M}$ and the Cauchy data are chosen to be real, then $\psi(t)$ is real for all times t. We also remark that (3.5) extends to

$$\psi(t) = W(t-s)\partial_s\psi(s) - \big(\partial_s W(t-s)\big)\psi(s),$$

valid for all $t, s \in \mathbb{R}$.

Similar arguments lead to the following slightly more general result which will be useful below:

Proposition 9. *Suppose that ψ is defined by Eq. (3.5) with Cauchy data $(\psi_0, \dot{\psi}_0)$.*

(a) *If $(\psi_0, \dot{\psi}_0) \in H^{n+l,n+l-1}_{\mathcal{M},m^2}$, $n \in \mathbb{N}_0$, $l \in \mathbb{N}$, then $\partial_t^n\psi(t)$, is of the form (3.5) with $\partial_t^n\psi(t) \in H^l_{\mathcal{M},m^2}$ for all t, and $\partial_t^n\psi(t)$ is l times strongly continuously differentiable in t. If $l \geqslant 2$, then $\partial_t^n\psi(t)$ is a solution of the wave equation with Cauchy data $(\Delta^k\psi_0, \Delta^k\dot{\psi}_0)$ if $n = 2k$, and with $(\Delta^k\dot{\psi}_0, \Delta^{k+1}\psi_0)$, respectively, if $n = 2k+1$. Furthermore, if $l \geqslant 3$, then the set $[\partial_t^n\psi(t)]$ of boundary values of $\partial_t^n\psi(t)$ at the vertices of $\mathcal{G}$ exists.*

(b) *If $(\psi_0, \dot{\psi}_0) \in H^{n+2,n+1}_{\mathcal{M},m^2}$, $n \in \mathbb{N}_0$, then $(-\Delta_{\mathcal{M}} + m^2)^{n/2}\psi(t)$ is a solution of the wave equation of the form (3.5) with Cauchy data $((-\Delta_{\mathcal{M}} + m^2)^{n/2}\psi_0, (-\Delta + m^2)^{n/2}\dot{\psi}_0)$. If $(\psi_0, \dot{\psi}_0) \in H^{n+3,n+2}_{\mathcal{M},m^2}$, $n \in \mathbb{N}_0$, then the boundary values $[(-\Delta + m^2)^{n/2}\psi(t)]$ are well-defined for all t.*

With (3.1) and (3.3), the corresponding estimates in terms of the Cauchy data are given by

Proposition 10. *The following* a priori *estimates are valid for $\psi(t)$, as defined by (3.5) with Cauchy data $(\psi_0, \dot{\psi}_0)$:*

(a) *Suppose that $(\psi_0, \dot{\psi}_0) \in H^{n+2k,n+2k}_{\mathcal{M},m^2}(\mathcal{G})$, $k, n \in \mathbb{N}_0$. Then for all t*

$$\big\|\partial_t^{2k}\psi(t)\big\|_{\mathcal{M},n} \leqslant \rho_{\mathcal{M}}(t)\big(\big\|\Delta^k_{\mathcal{M}}\psi_0\big\|_{\mathcal{M},n} + |t|\big\|\Delta^k_{\mathcal{M}}\dot{\psi}_0\big\|_{\mathcal{M},n}\big)$$

holds.

(b) *Suppose that $(\psi_0, \dot{\psi}_0) \in H^{n+2k+2,n+2k}_{\mathcal{M},m^2}(\mathcal{G})$, $k, n \in \mathbb{N}_0$. Then for all t*

$$\big\|\partial_t^{2k+1}\psi(t)\big\|_{\mathcal{M},n} \leqslant \rho_{\mathcal{M}}(t)\big(\big\|\Delta^k_{\mathcal{M}}\dot{\psi}_0\big\|_{\mathcal{M},n} + |t|\big\|\Delta^{k+1}_{\mathcal{M}}\psi_0\big\|_{\mathcal{M},n}\big)$$

holds.

Assume that $(\psi_0, \dot{\psi}_0) \in H^{2,2}_{\mathcal{M},m^2}$, so that by Proposition 9 $\partial_t\psi(t)$ is strongly continuously differentiable in t. Hence we can write

$$(\partial_t\psi)(t_1) - (\partial_t\psi)(t_2) = \int_{t_2}^{t_1} \big(\partial_t^2\psi\big)(s)\,ds,$$

as a relation in the Hilbert space $L^2(\mathcal{G})$, where the last integral is a Bochner integral. With the estimates given in Proposition 10 we therefore find

$$\begin{aligned}&\big\|(\partial_t\psi)(t_1)-(\partial_t\psi)(t_2)\big\|\\&\quad\leqslant|t_1-t_2|\max\big(\rho_{\mathcal{M}}(t_1),\rho_{\mathcal{M}}(t_2)\big)\big(\|\Delta_{\mathcal{M}}\psi_0\|+\max\big(|t_1|,|t_2|\big)\|\Delta_{\mathcal{M}}\dot{\psi}_0\|\big),\end{aligned}$$

and by hypothesis the last two norms are finite. This argument is readily generalized to provide the following result

Proposition 11. *Suppose that $\psi(t)$ is defined by (3.5) with Cauchy data $(\psi_0,\dot{\psi}_0)$.*

(a) *Assume that $(\psi_0,\dot{\psi}_0)\in H^{n+2k+2,n+2k}_{\mathcal{M},m^2}(\mathcal{G})$, $k,n\in\mathbb{N}_0$. Then for all t_1, t_2*

$$\begin{aligned}&\big\|\big(\partial_t^{2k}\psi\big)(t_1)-\big(\partial_t^{2k}\psi\big)(t_2)\big\|_{\mathcal{M},n}\\&\quad\leqslant|t_1-t_2|\max\big(\rho_{\mathcal{M}}(t_1),\rho_{\mathcal{M}}(t_2)\big)\big(\big\|\Delta_{\mathcal{M}}^k\dot{\psi}_0\big\|_{\mathcal{M},n}+\max\big(|t_1|,|t_2|\big)\big\|\Delta_{\mathcal{M}}^{k+1}\psi_0\big\|_{\mathcal{M},n}\big)\end{aligned}$$

holds true.

(b) *Assume that $(\psi_0,\dot{\psi}_0)\in H^{n+2k+2,n+2k+2}_{\mathcal{M},m^2}(\mathcal{G})$, $k,n\in\mathbb{N}_0$, then for all t_1, t_2*

$$\begin{aligned}&\big\|\big(\partial_t^{2k+1}\psi\big)(t_1)-\big(\partial_t^{2k+1}\psi\big)(t_2)\big\|_{\mathcal{M},n}\\&\quad\leqslant|t_1-t_2|\max\big(\rho_{\mathcal{M}}(t_1),\rho_{\mathcal{M}}(t_2)\big)\big(\big\|\Delta_{\mathcal{M}}^{k+1}\psi_0\big\|_{\mathcal{M},n}+\max\big(|t_1|,|t_2|\big)\big\|\Delta_{\mathcal{M}}^{k+1}\dot{\psi}_0\big\|_{\mathcal{M},n}\big)\end{aligned}$$

holds.

Next we consider the case where the boundary conditions defined by $\mathcal{M}$ are such that $-\Delta_{\mathcal{M}}$ is non-negative. Recall that in this case we make the choice $m^2=0$, and that $\rho_{\mathcal{M}}(t)$ is equal to 1 for all $t\in\mathbb{R}$. Moreover, we remark that then $\sqrt{-\Delta_{\mathcal{M}}}$ is a well-defined self-adjoint operator with domain $H^1_{\mathcal{M}}(\mathcal{G})$. Let ψ be defined as in (3.5), and let $k\in\mathbb{N}_0$. Then we get from (3.4) for all t

$$\partial_t^{2k}\psi(t)=\cos\big(\sqrt{-\Delta_{\mathcal{M}}}t\big)\Delta_{\mathcal{M}}^k\psi_0+\sin\big(\sqrt{-\Delta_{\mathcal{M}}}t\big)\Delta_{\mathcal{M}}^{(2k-1)/2}\dot{\psi}_0, \tag{3.7a}$$

$$\partial_t^{2k+1}\psi(t)=\sin\big(\sqrt{-\Delta_{\mathcal{M}}}t\big)\Delta_{\mathcal{M}}^{(2k+1)/2}\psi_0+\cos\big(\sqrt{-\Delta_{\mathcal{M}}}t\big)\Delta_{\mathcal{M}}^k\dot{\psi}_0, \tag{3.7b}$$

for the Cauchy data $(\psi_0,\dot{\psi}_0)$ in Sobolev spaces of sufficiently high degree. Hence we find the following result.

Corollary 12. *Suppose that the boundary conditions defined by $\mathcal{M}$ are such that $\Omega_{\mathcal{M}}\geqslant 0$, and hence $-\Delta_{\mathcal{M}}$ is non-negative. Assume furthermore that ψ is defined by (3.5) with Cauchy data $(\psi_0,\dot{\psi}_0)$. If $(\psi_0,\dot{\psi}_0)\in H^{n,n-1}_{\mathcal{M}}(\mathcal{G})$, $n\in\mathbb{N}$, then $\psi(t)$ is n times strongly continuously differentiable in t.*

The formulae (3.7) lead to alternative estimates as compared to those which we obtain directly from Proposition 10 for $m^2=0$. They are given in the next proposition, where we also combine them with the estimates of Proposition 10.

Corollary 13. *Suppose that $\mathcal{M}$ and ψ are as in the hypothesis of Corollary 12.*

(a) *For all $\psi_0, \dot\psi_0 \in L^2(\mathcal{G})$, the following* a priori *estimate is valid*

$$\|\psi(t)\| \leqslant \|\psi_0\| + |t|\|\dot\psi_0\|.$$

If $(\psi_0, \dot\psi_0) \in H^{n+k,n+k-1}$, $n, k \in \mathbb{N}_0$, with $n+k \geqslant 1$, then

$$\|\partial_t^k \psi(t)\|_{\mathcal{M},n} \leqslant \|\psi_0\|_{\mathcal{M},n+k} + \|\dot\psi_0\|_{\mathcal{M},n+k-1}$$

holds true for all t.

(b) *Assume that $(\psi_0, \dot\psi_0) \in H_{\mathcal{M}}^{n+2l,n+2l}$, $l \in \mathbb{N}$, $n \in \mathbb{N}_0$, then*

$$\|\partial_t^{2l} \psi(t)\|_{\mathcal{M},n} \leqslant \|\psi_0\|_{\mathcal{M},n+2l} + \min\big(|t|\|\dot\psi_0\|_{\mathcal{M},n+2l}, \|\dot\psi_0\|_{\mathcal{M},n+2l-1}\big)$$

is valid for all t. If $(\psi_0, \dot\psi_0) \in H_{\mathcal{M}}^{n+2l+2,n+2l}$, $l \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, then

$$\|\partial_t^{2l+1} \psi(t)\|_{\mathcal{M},n} \leqslant \|\dot\psi_0\|_{\mathcal{M},n+2l} + \min\big(|t|\|\psi_0\|_{\mathcal{M},n+2l+2}, \|\psi_0\|_{\mathcal{M},n+2l+1}\big)$$

holds for all t.

For the analogue of Corollary 13 in the case that $-\Delta_{\mathcal{M}} \geqslant 0$ we only give the form of the estimates as based on Eqs. (3.7):

Corollary 14. *Suppose that $\mathcal{M}$ and ψ are as in the hypothesis of Corollary 12. Assume that $(\psi_0, \dot\psi_0) \in H_{\mathcal{M}}^{n+k+1,n+k}(\mathcal{G})$, $k, n \in \mathbb{N}_0$. Then*

$$\|\big(\partial_t^k \psi\big)(t_1) - \big(\partial_t^k \psi\big)(t_2)\|_{\mathcal{M},n} \leqslant |t_1 - t_2|\big(\|\psi_0\|_{\mathcal{M},n+k+1} + \|\dot\psi_0\|_{\mathcal{M},n+k}\big)$$

holds true for all t_1, t_2.

Our discussion so far may not be specifically restricted to the context of metric graphs and self-adjoint Laplacians defined there. We could instead have considered any manifold with a self-adjoint Laplacian Δ there, for which $-\Delta$ is bounded below and which therefore defines a wave operator. We would have obtained the same type of estimates. From now on, however, the specific one-dimensional situation enters. We continue to consider the case where $\mathcal{M}$ is such that $-\Delta_{\mathcal{M}} \geqslant 0$. Let $f^{(j)}$, $j \in \mathbb{N}$ denote the j-th spatial derivative of any function f on $\mathcal{G}$ for which this derivative exists (in the sense of the derivative of a function or in the L^2-sense). One easily verifies that $\psi^{(2n)} = \Delta_{\mathcal{M}}^n \psi$ holds such that relation (2.6) extends to $\|\psi^{(2n)}\| = \|\Delta_{\mathcal{M}}^n \psi\|$ for all $\psi \in \mathcal{D}(\Delta_{\mathcal{M}}^n)$, while (2.9) extends to $\|\psi^{(2n+1)}\| \leqslant \|(\sqrt{-\Delta_{\mathcal{M}}})^{2n+1}\psi\|$ for all $\psi \in \mathcal{D}((\sqrt{-\Delta_{\mathcal{M}}})^{2n+1})$. Similarly Corollary 5 provides the following

Corollary 15. *Suppose that $\mathcal{M}$ and ψ are as in the hypothesis of Corollary 12.*

(a) *If* $(\psi_0, \dot{\psi}_0) \in H_{\mathcal{M}}^{n+j+k,n+j+k-1}(\mathcal{G})$, $j, k, n \in \mathbb{N}_0$, *with* $n + k + j \geqslant 1$, *then*

$$\left\| \left(\partial_t^k \psi(t)\right)^{(j)} \right\|_{\mathcal{M},n} \leqslant \|\psi_0\|_{\mathcal{M},n+j+k} + \|\dot{\psi}_0\|_{\mathcal{M},n+j+k-1}$$

is valid for all t.

(b) *If* $(\psi_0, \dot{\psi}_0) \in H_{\mathcal{M}}^{n+j+k+1,n+j+k}(\mathcal{G})$, $j, k, n \in \mathbb{N}_0$, *then*

$$\left\| \left((\partial_t^k \psi)(t_1)^{(j)} - \left(\partial_t^k \psi\right)(t_2)^{(j)}\right) \right\|_{\mathcal{M},n} \leqslant |t_1 - t_2| \left(\|\psi_0\|_{\mathcal{M},n+j+k+1} + \|\dot{\psi}_0\|_{\mathcal{M},n+j+k}\right)$$

holds true for all t.

We return to the general case, i.e., we do *not* assume that $\mathcal{M}$ is such that $-\Delta_{\mathcal{M}}$ is non-negative except where otherwise stated.

To establish uniqueness of the solution (3.5) for given Cauchy data, we introduce the energy functional. For any solution $\varphi(t)$ of the wave equation with t in a time interval $[-T, T]$, say, set

$$\begin{aligned} E_{\mathcal{M}}\left(\varphi(t)\right) &= \frac{1}{2}\left\|\partial_t \varphi(t)\right\|_{\mathcal{G}}^2 + \frac{1}{2}\left\langle \varphi(t), -\Delta_{\mathcal{M}}\varphi(t)\right\rangle \\ &= \frac{1}{2}\left\|\partial_t \varphi(t)\right\|_{\mathcal{G}}^2 + \frac{1}{2}\left\|\sqrt{-\Delta_{\mathcal{M}} + m^2}\varphi(t)\right\|_{\mathcal{G}}^2 - \frac{m^2}{2}\left\|\varphi(t)\right\|_{\mathcal{G}}^2 \end{aligned}$$

which is finite provided $\varphi(t) \in H^1_{\mathcal{M},m^2}(\mathcal{G})$ and $\varphi(t)$ is strongly differentiable in t for all $t \in [-T, T]$. $\langle \varphi(t), -\Delta_{\mathcal{M}}\varphi(t)\rangle$ is understood in the sense of quadratic forms. The factor $1/2$ is inserted in order to conform with the standard normalization convention.

In particular $E_{\mathcal{M}}(\varphi(t))$ is finite for all t when $\varphi(t) = \psi(t)$ with $\psi(t)$ as given by (3.5) with Cauchy data $(\psi_0, \dot{\psi}_0) \in H^{1,0}_{\mathcal{M},m^2}(\mathcal{G})$.

Proposition 16. *Let* φ *be any solution of the wave equation* (3.6) *in the time interval* $[-T, T]$ *and having the following properties. For all* $t \in [-T, T]$

(a) $\varphi(t) \in H^2_{\mathcal{M},m^2}(\mathcal{G})$,
(b) $\varphi(t)$ *is three times strongly differentiable in* t,
(c) $\partial_t \varphi(t) \in H^2_{\mathcal{M},m^2}(\mathcal{G})$,
(d) $\partial_t \varphi(t)$ *also satisfies the wave equation.*

Then the energy functional $E_{\mathcal{M}}(\varphi(t))$, $t \in [-T, T]$, *is time independent. In addition, if* $\mathcal{M}$ *is such that* $-\Delta_{\mathcal{M}} \geqslant 0$ *holds, then the energy functional* $E_{\mathcal{M}}(\varphi(t))$, $t \in [-T, T]$, *is non-negative and vanishes if and only if both* $-\Delta_{\mathcal{M}}\varphi(t)$ *and* $\partial_t \varphi(t)$ *vanish for all times* $t \in [-T, T]$.

Again observe that for $\varphi(t) = \psi(t)$ of the form (3.5) with Cauchy data $(\psi_0, \dot{\psi}_0) \in H^{3,2}_{\mathcal{M},m^2}(\mathcal{G})$ the assumptions of Proposition 16 are satisfied.

Proof. By the assumptions we are free to differentiate $E_{\mathcal{M}}(\varphi(t))$ with respect to the time $t \in \mathbb{R}$. We claim that the relation

$$\partial_t\left(-\Delta_{\mathcal{M}}\varphi(t)\right) = -\Delta_{\mathcal{M}}\partial_t \varphi(t) \tag{3.8}$$

is valid. In fact, by assumption

$$\partial_t\bigl(-\Delta_{\mathcal{M}}\varphi(t)\bigr) = -\partial_t\bigl(\partial_t^2\varphi(t)\bigr) = -\partial_t^3\varphi(t) = -\partial_t^2\bigl(\partial_t\varphi(t)\bigr) = -\Delta_{\mathcal{M}}\partial_t\varphi(t)$$

holds. Another way to obtain this is to observe that $-\Delta_{\mathcal{M}}$ is a linear operator and therefore (3.8) holds. Thus by standard calculations the time derivative of $E_{\mathcal{M}}(\varphi(t))$ vanishes, thus establishing the first claim. As for the second claim, assume that $E_{\mathcal{M}}(\varphi(t)) = 0$ for all t. But this implies $\partial_t\varphi(t) = 0$ and $\sqrt{-\Delta_{\mathcal{M}}}\varphi(t) = 0$, which in turn gives $-\Delta_{\mathcal{M}}\psi(t) = 0$. The converse is trivial. □

Theorem 17. *Let $\mathcal{M}$ be such that $-\Delta_{\mathcal{M}} \geqslant 0$ and such that 0 is not an eigenvalue of $-\Delta_{\mathcal{M}}$. Let $\varphi_1(t)$ and $\varphi_2(t)$ be two solutions of the wave equation for $t \in [-T, T]$ satisfying the assumptions of Proposition 16 and with the same initial values,*

$$\varphi_1(t=0) = \varphi_2(t=0), \qquad \partial_t\varphi_1(t=0) = \partial_t\varphi_2(t=0).$$

Then $\varphi_1(t) = \varphi_2(t)$ holds for all $t \in [-T, T]$. In particular ψ as given by (3.5) is the unique solution of the wave equation for given Cauchy data $(\psi_0, \dot{\psi}_0) \in H^{3,2}_{\mathcal{M},m^2}(\mathcal{G})$.

For a given metric graph, necessary and sufficient conditions on $\mathcal{M}$ for $-\Delta_{\mathcal{M}}$ to have 0 as an eigenvalue are given in [14], see also Corollary 4. If ψ^0 is such an eigenfunction, it has to be constant on each edge and in particular zero on each external edge. Also $\psi^0(t)$ as given by (3.5) with Cauchy data $(\psi^0, \dot{\psi} = 0)$ satisfies $\psi^0(t) = \psi^0$ for all $t \in \mathbb{R}$.

Proof of Theorem 17. Standard and well known arguments can now be used. Indeed, $\varphi_1(t) - \varphi_2(t)$ is a solution of the wave equation with vanishing initial data and we can use the previous proposition. □

In order to establish finite propagation speed, we introduce a local form of the energy functional. As a motivation we use (2.8) to rewrite the energy functional as

$$E_{\mathcal{M}}\bigl(\psi(t)\bigr) = \frac{1}{2}\bigl(\bigl\|\partial_t\psi(t)\bigr\|_{\mathcal{G}}^2 + \bigl\|\psi(t)'\bigr\|_{\mathcal{G}}^2 + \bigl\langle\bigl[\psi(t)\bigr], \Omega_{\mathcal{M}}\bigl[\psi(t)\bigr]\bigr\rangle_{^d\mathcal{K}}\bigr), \tag{3.9}$$

which is finite for all t provided the Cauchy data $(\psi_0, \dot{\psi}_0)$ are such that $\psi_0 \in H^2_{\mathcal{M},m^2}(\mathcal{G})$, $\dot{\psi}_0 \in H^1_{\mathcal{M},m^2}(\mathcal{G})$, cf. the remarks after Proposition 3 and Lemma 8.

The first two terms on the right-hand side form the energy functional for solutions of the wave equation on smooth manifolds, see e.g. [2,4,17,18]. So it is the last term which is special for the present context of metric graphs, which are singular manifolds. Of these three terms it is the only one, in which the boundary condition $\mathcal{M}$ enters and, as we shall see, in a manageable way. For the remainder of this section we assume that $\mathcal{M}$ is such that $\Omega_{\mathcal{M}} \geqslant 0$, and hence also $-\Delta_{\mathcal{M}} \geqslant 0$, as well as $(\psi_0, \dot{\psi}_0) \in H^{4,3}_{\mathcal{M}}(\mathcal{G})$. Then Lemma 27 in Appendix A entails that $\|\psi(t)'\|^2$ is differentiable in t. Since $\langle\psi(t), -\Delta_{\mathcal{M}}\psi(t)\rangle = \langle\psi(t), -\partial_t^2\psi(t)\rangle$ is also differentiable in t (see Proposition 9), we conclude that

$$\bigl\langle\bigl[\psi(t)\bigr], \Omega_{\mathcal{M}}\bigl[\psi(t)\bigr]\bigr\rangle_{^d\mathcal{K}} = \bigl\langle\psi(t), -\Delta_{\mathcal{M}}\psi(t)\bigr\rangle - \bigl\|\psi(t)'\bigr\|_{\mathcal{G}}^2$$

is differentiable with respect to t too.

Actually, more is valid and will be used, namely we also have differentiability of the boundary values $[\psi(t)]$ themselves.

Lemma 18. *Assume the boundary condition $\mathcal{M}$ is such that $\Omega_{\mathcal{M}} \geqslant 0$ and hence also $-\Delta_{\mathcal{M}} \geqslant 0$ is valid. For the Cauchy data $(\psi_0, \dot{\psi}_0)$ in $H^{4,3}_{\mathcal{M}}(\mathcal{G})$ the boundary value $[\psi(t)]$ is continuously differentiable in t and*

$$\partial_t\bigl[\psi(t)\bigr] = \bigl[\partial_t \psi(t)\bigr] \tag{3.10}$$

holds.

Observe in this context that since $\psi(t) \in \operatorname{Dom}(-\Delta_{\mathcal{M}})$ is valid for all t, the relation $\mathcal{P}_{\mathcal{M}}[\psi(t)] = [\psi(t)]$ holds for all t which upon differentiation gives

$$\mathcal{P}_{\mathcal{M}}\bigl[\partial_t \psi(t)\bigr] = \bigl[\partial_t \psi(t)\bigr]. \tag{3.11}$$

Alternatively (3.11) follows from $\partial_t \psi(t) \in \operatorname{Dom}(-\Delta_{\mathcal{M}})$, which in turn follows from the assumptions on the Cauchy data. The proof of Lemma 18 is based on Sobolev estimates in conjunction with Corollary 15 and will be given in Appendix A.

4. Finite propagation speed

The form (3.9) allows us to introduce a local energy functional. Fix a point $p \in \mathcal{G}$ and a time $t_0 \in \mathbb{R}$. For any ψ of the form (3.5) with Cauchy data $(\psi_0, \dot{\psi}_0) \in H^{2,1}_{\mathcal{M},m^2}(\mathcal{G})$ and $0 \leqslant t \leqslant t_0$, the time dependent *local energy functional* is defined as

$$e(t) = \frac{1}{2}\bigl(\bigl\|\partial_t \psi(t)\bigr\|^2_{B(p,t_0-t)} + \bigl\|\psi(t)'\bigr\|^2_{B(p,t_0-t)} + \bigl\langle\bigl[\psi(t)\bigr], \Omega_t\bigl[\psi(t)\bigr]\bigr\rangle_{d\mathcal{K}}\bigr), \tag{4.1}$$

where

$$\Omega_t = \Omega_{\mathcal{M}, B(p,t_0-t)\cap\mathcal{V}} = P_t \Omega_{\mathcal{M}} P_t = P_t \Omega_{\mathcal{M}} = \Omega_{\mathcal{M}} P_t,$$

cf. (2.13), with

$$P_t = \sum_{v \in B(p,t_0-t)\cap\mathcal{V}} P_v$$

satisfying $P_t \leqslant P_{t'}$ for $t' \leqslant t$. Ω_t is piecewise constant in t with possible jumps at $\mathcal{T}(p)$. Observe that all terms on the right-hand side of (4.1) are finite: By Proposition 9, the hypothesis $(\psi_0, \dot{\psi}_0) \in H^{2,1}_{\mathcal{M},m^2}(\mathcal{G})$ implies that $\psi(t) \in H^2_{\mathcal{M},m^2}(\mathcal{G}) = \operatorname{Dom}(\Delta_{\mathcal{M}})$ for all $t \in \mathbb{R}$. Thus, on every edge of $\mathcal{G}$, $\psi(t)$ and $\psi(t)'$ are continuous functions, and in particular their boundary values $[\psi(t)]$ at the vertices of $\mathcal{G}$ are well-defined and finite.

The initial value of $e(t)$ can be expressed in terms of the Cauchy data themselves as

$$e(t=0) = \frac{1}{2}\bigl(\|\dot{\psi}_0\|^2_{B(p,t_0)} + \bigl\|\psi_0'\bigr\|^2_{B(p,t_0)} + \bigl\langle[\psi_0], \Omega_{t=0}[\psi_0]\bigr\rangle_{d\mathcal{K}}\bigr).$$

We will be interested in the situation when the condition $\Omega_{\mathcal{M}} \geqslant 0$ is satisfied and then obviously $0 \leqslant e(t)$ for all t. Also by Lemma 7

$$\Omega_t \leqslant \Omega_{t'}, \quad t' \leqslant t, \tag{4.2}$$

is valid. Now, for $e(t)$ to vanish when $0 \leqslant \Omega_{\mathcal{M}}$, it is necessary that both $\partial_t \psi(t)$ and $\psi(t)'$ vanish on $B(p, t_0 - t)$. In particular $\psi(t)$ is then piecewise constant, that is $\psi_i(t)$ is constant on each $B(p, t_0 - t) \cap I_i$, $i \in \mathcal{E} \cup \mathcal{I}$, which is a connected set. In the case where actually $\Omega_{\mathcal{M}} > 0$ holds, for $e(t)$ to vanish it is necessary and sufficient that both $\partial_t \psi(t)$ and $\psi(t)'$ vanish on $B(p, t_0 - t)$ and that $P_t[\psi(t)] = 0$.

We want to show that $e(t)$ is non-increasing in t. To establish this we need a couple of lemmas. The first one is a local version of Proposition 3. For its formulation we need an adaption of the familiar notion of a normal derivative to the present context.

Definition 19. Assume $0 < t_0 - t \notin \mathcal{T}(p)$ with $p \cong (k, y)$. The *inward normal derivative* of ψ at $q \in \partial B(p, t_0 - t)$ with coordinate $q \cong (i, x)$ $(0 < x < a_i, i \in \mathcal{E} \cup \mathcal{I})$ is defined as

$$\partial_{\mathbf{n}} \psi(q) = \begin{cases} \psi_i'(x), & \text{if } k = i,\ x < y,\ \text{or if } i \neq k \text{ and } [x, a_i] \subset B(p, t_0 - t), \\ -\psi_i'(x), & \text{if } k = i,\ y < x,\ \text{or if } i \neq k \text{ and } [0, x] \subset B(p, t_0 - t). \end{cases}$$

The sign convention is made to conform with the sign convention in the definition (2.3) of $\underline{\psi}'$ and hence of $[\psi]$. As an example consider the case $k = i \in \mathcal{E}$, again with $p = (k, y)$ and in addition t so close to t_0 that $0 < t_0 - t < y$. Then $B(p, t_0 - t)$ is an interval on $I_k \cong [0, \infty)$ of the form $[y - t_0 + t, y + t_0 - t]$ centered at y and of length $|B(p, t_0 - t)| = 2(t_0 - t)$. So $\partial B(p, t_0 - t)$ consists of the two points $(k, y - t_0 + t)$ and $(k, y + t_0 - t)$ such that $\partial_{\mathbf{n}} \psi(k, y - t_0 + t) = \psi_k'(y - t_0 + t)$ and $\partial_{\mathbf{n}} \psi(k, y + t_0 - t) = -\psi_k'(y + t_0 - t)$.

Lemma 20. *For every boundary condition $\mathcal{M}$, and any $t_0 - t \in \mathbb{R}_+ \setminus \mathcal{T}(p)$ the relation*

$$\langle \varphi, -\Delta_{\mathcal{M}} \psi \rangle_{B(p,t_0-t)} = \langle \varphi', \psi' \rangle_{B(p,t_0-t)} + \langle [\varphi], \Omega_t[\psi] \rangle_{d\mathcal{K}} + \sum_{q \in \partial B(p,t_0-t)} \overline{\varphi(q)} \partial_{\mathbf{n}} \psi(q) \tag{4.3}$$

is valid for any $\varphi, \psi \in \operatorname{Dom}(-\Delta_{\mathcal{M}})$.

Proof. Observe that by the remark after Proposition 3 or — in the case that $-\Delta_{\mathcal{M}} \geqslant 0$ — more easily by Corollary 5 both φ' and ψ' are elements in $L^2(\mathcal{G})$. Furthermore since $\operatorname{Dom}(-\Delta_{\mathcal{M}}) \subset \mathcal{D}$ all terms on the right-hand side of (4.3) are well-defined and finite. This allows us to perform an integration by parts and to use Green's identity. Firstly there are boundary contributions at those vertices, which are contained in $B(p, t_0 - t)$, and secondly at points of the boundary $\partial B(p, t_0 - t)$ giving

$$\begin{aligned} &\langle \varphi, -\Delta_{\mathcal{M}} \psi \rangle_{B(p,t_0-t)} \\ &\quad = \langle \varphi', \psi' \rangle_{B(p,t_0-t)} + \sum_{v \in B(p,t_0-t)} \langle {}^d Q_v[\varphi], \Omega^d Q_v[\psi] \rangle_{d\mathcal{K}} + \sum_{q \in \partial B(p,t_0-t)} \overline{\varphi(q)} \partial_{\mathbf{n}} \psi(q). \end{aligned}$$

Now we insert $P_{\mathcal{M}}[\varphi] = [\varphi]$ and $P_{\mathcal{M}}[\psi] = [\psi]$, valid due to the assumption $\varphi, \psi \in \operatorname{Dom}(-\Delta_{\mathcal{M}})$, into the second term. Using in addition (2.12) we obtain

$$\sum_{v\in B(p,t_0-t)} \langle {}^d Q_v[\varphi], \Omega^d Q_v[\psi]\rangle_{d\mathcal{K}} = \sum_{v\in B(p,t_0-t)} \langle [\varphi], P_{\mathcal{M}}{}^d Q_v \Omega^d Q_v P_{\mathcal{M}}[\psi]\rangle_{d\mathcal{K}}$$
$$= \langle [\varphi], \Omega_t[\psi]\rangle_{d\mathcal{K}}. \qquad \square$$

Proposition 21. *Assume that the boundary condition $\mathcal{M}$ is such that $\Omega_{\mathcal{M}} \geqslant 0$ and hence $-\Delta_{\mathcal{M}} \geqslant 0$. Also let the Cauchy data $(\psi_0, \dot\psi_0)$ be such that $\psi_0 \in H^4_{\mathcal{M}}(\mathcal{G})$ and $\dot\psi_0 \in H^3_{\mathcal{M}}(\mathcal{G})$. Then $e(t)$ is differentiable at all points t with $t_0 - t \in \mathbb{R}_+ \setminus \mathcal{T}(p)$ and satisfies $\partial_t e(t) \leqslant 0$ there.*

Proof. We differentiate $e(t)$ under the assumption on t that $\partial B(p, t_0 - t) \cap \mathcal{V} = \emptyset$ which in particular means that Ω_s is constant for all s close to t. We use (3.10) and obtain

$$\begin{aligned}
\partial_t e(t) &= \frac{1}{2}\langle \partial_t^2\psi(t), \partial_t\psi(t)\rangle_{B(p,t_0-t)} + \frac{1}{2}\langle \partial_t\psi(t), \partial_t^2\psi(t)\rangle_{B(p,t_0-t)} \\
&\quad + \frac{1}{2}\langle \partial_t\psi(t)', \psi(t)'\rangle_{B(p,t_0-t)} + \frac{1}{2}\langle \psi(t)', \partial_t\psi(t)'\rangle_{B(p,t_0-t)} \\
&\quad - \frac{1}{2}\sum_{q\in\partial B(p,t_0-t)} \left(\left|\partial_t\psi(t,q)\right|^2 + \left|\psi'(t,q)\right|^2\right) \\
&\quad + \frac{1}{2}\langle [\partial_t\psi(t)], \Omega_t[\psi(t)]\rangle_{d\mathcal{K}} + \frac{1}{2}\langle [\psi(t)], \Omega_t[\partial_t\psi(t)]\rangle_{d\mathcal{K}}
\end{aligned}$$

with the abbreviation $\psi(t,q) = \psi(t)(q)$. In the next step we first invoke the wave equation (3.6) for the first two terms on the right-hand side, and then use Lemma 20. This gives

$$\begin{aligned}
\partial_t e(t) &= -\frac{1}{2}\sum_{q\in\partial B(p,t_0-t)} \Big(\left|\partial_t\psi(t,q)\right|^2 + \left|\psi(t,q)'\right|^2 \\
&\qquad + \overline{\partial_t\psi(t,q)}\,\partial_{\mathbf{n}}\psi(t,q) + \overline{\partial_{\mathbf{n}}\psi(t,q)}\partial_t\psi(t,q)\Big) \\
&= -\frac{1}{2}\sum_{q\in\partial B(p,t_0-t)} \left|\partial_t\psi(t,q) + \partial_{\mathbf{n}}\psi(t,q)\right|^2 \leqslant 0,
\end{aligned} \tag{4.4}$$

and the proof is finished. $\square$

Next we look at what happens when $t_0 - t \in \mathcal{T}(p)$. As a motivation for our further procedure, we show why relation (4.4) fails when $\partial B(p, t_0 - t)$ contains coinciding points. To simplify the discussion we assume there is only one coinciding point $q \in \partial B(p, t_0 - t)$ and that $\partial B(p, t_0 - t) \cap \mathcal{V} = \emptyset$. With the notation used in Definition 1, for all $s > t$ sufficiently close to t there will be a contribution to $\partial_s e(s)$ of the form

$$\begin{aligned}
&-\frac{1}{2}\overline{\partial_t\psi\big(s, q_l(t_0 - s)\big)}\partial_{\mathbf{n}}\psi\big(s, q_l(t_0 - s)\big) - \frac{1}{2}\overline{\partial_t\psi\big(s, q_r(t_0 - s)\big)}\partial_{\mathbf{n}}\psi\big(s, q_r(s)\big) \\
&\quad - \frac{1}{2}\overline{\partial_{\mathbf{n}}\psi\big(s, q_l(t_0 - s)\big)}\partial_t\psi\big(s, q_l(t_0 - s)\big) - \frac{1}{2}\overline{\partial_{\mathbf{n}}\psi\big(s, q_r(t_0 - s)\big)}\partial_t\psi\big(s, q_r(t_0 - s)\big).
\end{aligned} \tag{4.5}$$

Since $\lim_{s\downarrow t} q_l(t_0 - s) = \lim_{s\downarrow t} q_r(t_0 - s) = q$, by continuity

$$\lim_{s\downarrow t} \partial_t \psi\bigl(s, q_l(t_0 - s)\bigr) = \lim_{s\downarrow t} \partial_t \psi\bigl(s, q_r(t_0 - s)\bigr)$$

while

$$\lim_{s\downarrow t} \partial_{\mathbf{n}} \psi\bigl(q_l(s), s\bigr) = -\lim_{s\downarrow t} \partial_{\mathbf{n}} \psi\bigl(q_r(s), s\bigr)$$

so the terms in (4.5) cancel pairwise when s decreases to t.

Proposition 22. *Assume the boundary condition $\mathcal{M}$ is such that $\Omega_{\mathcal{M}} \geqslant 0$ and hence also $-\Delta_{\mathcal{M}} \geqslant 0$. Also let the Cauchy data $(\psi_0, \dot{\psi}_0)$ be such that $\psi_0 \in H^4_{\mathcal{M}}(\mathcal{G})$ and $\dot{\psi}_0 \in H^3_{\mathcal{M}}(\mathcal{G})$. The following relation holds*

$$\lim_{s\uparrow t} e(s) = e(t) \geqslant \lim_{s\downarrow t} e(s)$$

for all $t_0 - t \in \mathcal{T}(p)$.

For the proof we need

Lemma 23. *Under the assumptions of Proposition 22 on $\mathcal{M}$ and the Cauchy data, the map*

$$t \mapsto \frac{1}{2}\bigl(\bigl\|\partial_t \psi(t)\bigr\|^2_{B(p,t_0-t)} + \bigl\|\psi(t)'\bigr\|^2_{B(p,t_0-t)}\bigr)$$

is continuous in $t \in \mathbb{R}_+$.

The proof of this lemma will be given in Appendix A and is based on Lemma 26, whose proof is also given there.

Under the assumption $\Omega_{\mathcal{M}} \geqslant 0$ and by (4.2)

$$\lim_{s\uparrow t} \Omega_s \geqslant \lim_{s\downarrow t} \Omega_s. \tag{4.6}$$

More explicitly

$$\lim_{s\uparrow t} \Omega_s = \Omega_t = \sum_{v\in\partial B(t_0-t)\cap\mathcal{V}} \Omega_v + \lim_{s\downarrow t} \Omega_s \geqslant \lim_{s\downarrow t} \Omega_s.$$

We combine (4.6) with Lemma 18 and conclude

$$\lim_{s\uparrow t} \bigl\langle\bigl[\psi(s)\bigr], \Omega_s\bigl[\psi(s)\bigr]\bigr\rangle_{d\mathcal{K}} = \bigl\langle\bigl[\psi(t)\bigr], \Omega_t\bigl[\psi(t)\bigr]\bigr\rangle_{d\mathcal{K}} \geqslant \lim_{s\downarrow t} \bigl\langle\bigl[\psi(s)\bigr], \Omega_s\bigl[\psi(s)\bigr]\bigr\rangle_{d\mathcal{K}}.$$

This result combined with Lemma 23 concludes the proof of Proposition 22. In turn Propositions 21 and 22 give the first part of

Theorem 24 *(Finite propagation speed). Assume the boundary condition $\mathcal{M}$ is such that $\Omega_{\mathcal{M}} \geqslant 0$ and hence $-\Delta_{\mathcal{M}} \geqslant 0$. For the Cauchy data $\psi_0 \in H^4_{\mathcal{M}}(\mathcal{G})$ and $\dot{\psi}_0 \in H^3_{\mathcal{M}}(\mathcal{G})$, let $\psi(t)$ be defined by (3.5). Fix a point p and a time $t_0 > 0$. Then $e(t)$ as defined by (4.1) is non-negative and non-increasing for $0 \leqslant t \leqslant t_0$. If ψ_0 and $\dot{\psi}_0$ both vanish on $B(p,t_0)$, then $\psi(t,q)$ vanishes on the cone*

$$\mathcal{C}(p,t_0) = \bigl\{(t,q) \bigm| 0 \leqslant t \leqslant t_0,\ d(q,p) \leqslant t_0 - t\bigr\} \subset \mathcal{G} \times \mathbb{R}_+$$

with vertex at (p,t_0).

Proof. The proof of the second part now uses standard arguments, see e.g. [4]. By assumption $e(t=0)=0$. Hence by the first part of the theorem, $e(t)=0$ for all $0 \leqslant t \leqslant t_0$. Thus $\partial_t \psi(t,q) = \psi'(t,q) = 0$ for $(t,q) \in \mathcal{C}(p,t_0)$. As a consequence

$$\psi(t,q) = \psi_0(q) + \int_0^t \partial_s \psi(s,q)\, ds = 0$$

for $(q,t) \in \mathcal{C}(p,t_0)$. $\square$

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Appendix A. Proof of Lemmas 18 and 23

Throughout the appendix $\mathcal{M}$ is chosen to be such that $\Omega_{\mathcal{M}} \geqslant 0$ holds. We recall the Sobolev inequality in 1 dimension, see e.g. [13, Theorem 8.5]. Any function f in the Sobolev space $H^1(\mathbb{R})$ is bounded and satisfies the estimate

$$\|f\|_\infty^2 \leqslant \frac{1}{2}\bigl(\|f\|^2_{L^2(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})}\bigr). \tag{A.1}$$

In order not to burden the notation, here and in what follows $\|\cdot\|_\infty$ will always denote the L^∞ norm while $\|\cdot\|$ is the L^2 norm in the respective context. The Sobolev inequality easily carries over to our context where $\mathbb{R}$ is replaced by $\mathcal{G}$

$$\|\psi\|_\infty^2 \leqslant \frac{1}{2}\bigl(\|\psi\|^2 + \|\psi'\|^2\bigr).$$

This inequality follows by simple arguments from (A.1), which are omitted here. More generally, for $\psi \in H^{j+1}_{\mathcal{M}}(\mathcal{G})$, $j \in \mathbb{N}_0$,

$$\|\psi^{(j)}\|_\infty^2 \leqslant \frac{1}{2}\bigl(\|\psi^{(j)}\|^2 + \|\psi^{(j+1)}\|^2\bigr)$$

holds. This inequality is now combined with Corollary 15 to obtain several estimates for $\psi(t)$, as defined by (3.6) with Cauchy data $(\psi_0, \dot{\psi}_0)$. For $n \in \mathbb{N}$, $(\psi_0, \dot{\psi}_0) \in H^{n,n-1}_{\mathcal{M}}(\mathcal{G})$ introduce

$$A_1(\psi_0,\dot\psi_0,t)=\bigl(\|\psi_0\|^2+\|\psi_0\|^2_{\mathcal{M},1}+\bigl(1+t^2\bigr)\|\dot\psi_0\|^2\bigr)^{1/2}$$

and for $n\geqslant 2$,

$$A_n(\psi_0\dot\psi_0)=\bigl(\|\psi_0\|^2_{\mathcal{M},n-1}+\|\psi_0\|^2_{\mathcal{M},n}+\|\dot\psi_0\|^2_{\mathcal{M},n-2}+\|\psi_0\|^2_{\mathcal{M},n-1}\bigr)^{1/2}.$$

We leave out the proof of the following lemma.

Lemma 25. *Suppose that $\mathcal{M}$ is such that $-\Delta_{\mathcal{M}}\geqslant 0$, and that $\psi(t)$ is given by* (3.5) *with Cauchy data $(\psi_0,\dot\psi_0)$. Then the following estimates hold true for all t, t_1, t_2:*

(a) *for $(\psi_0,\dot\psi_0)\in H^{1,0}_{\mathcal{M}}(\mathcal{G})$,*

$$\bigl\|\psi(t)\bigr\|_\infty\leqslant A_1(\psi_0,\dot\psi_0,t),\tag{A.2a}$$

and for $(\psi_0,\dot\psi_0)\in H^{j+k+1,j+k}_{\mathcal{M}}(\mathcal{G})$, $j,k\in\mathbb{N}_0$, with $j+k\geqslant 1$,

$$\bigl\|\bigl(\partial_t^k\psi\bigr)(t)^{(j)}\bigr\|_\infty\leqslant A_{j+k+1}(\psi_0,\dot\psi_0);\tag{A.2b}$$

(b) *for $(\psi_0,\dot\psi_0)\in H^{j+k+2,j+k+1}_{\mathcal{M}}(\mathcal{G})$, $j,k\in\mathbb{N}_0$,*

$$\bigl\|\bigl(\bigl(\partial_t^k\psi\bigr)(t_1)\bigr)^{(j)}-\bigl(\bigl(\partial_t^k\psi\bigr)(t_2)\bigr)^{(j)}\bigr\|_\infty\leqslant|t_1-t_2|A_{j+k+2}(\psi_0,\dot\psi_0).\tag{A.2c}$$

In the next step we will prove the continuity of

$$\bigl(\partial_t\psi(t)\bigr)_i'(x)=\frac{\partial}{\partial x}\frac{\partial}{\partial t}\psi_i(t,x)\tag{A.3}$$

in both $t\in\mathbb{R}$ and in $x\in I_i$, $i\in\mathcal{E}\cup\mathcal{I}$. This will enable us to establish both the existence of and the equality with the other mixed partial second derivative

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\psi_i(t,x).\tag{A.4}$$

To this end, we assume from now on that the Cauchy data $(\psi_0,\dot\psi_0)$ belong to $H^{4,3}_{\mathcal{M}}(\mathcal{G})$. So by Proposition 9 $\psi(t)\in H^4_{\mathcal{M}}(\mathcal{G})$ and $\partial_t\psi(t)\in H^3_{\mathcal{M}}(\mathcal{G})$ and therefore both have spatial derivatives up to third order in $L^2(\mathcal{G})$. As a consequence the restrictions of both to each edge have absolutely continuous spatial derivatives up to order two. Thus on every edge we may consider $\psi(t)^{(j)}$, $\partial_t\psi(t)^{(j)}$, $t\in\mathbb{R}$, $j=0,1,2$, as *bona fide* functions, and in particular their L^∞ norms equal their sup-norms.

Consider a fixed edge I_i of $\mathcal{G}$, $x_1,x_2\in I_i$, and let $j,k=0,1$. Then the mean value theorem together with inequality (A.2b) gives

$$\begin{aligned}\sup_{t\in\mathbb{R}}\bigl|\bigl(\partial_x^j\partial_t^k\psi\bigr)_i(t,x_1)-\bigl(\partial_x^j\partial_t^k\psi\bigr)_i(t,x_2)\bigr|&\leqslant|x_1-x_2|\sup_{t\in\mathbb{R}}\bigl\|\partial_t^k\psi(t)^{(j+1)}\bigr\|_\infty\\&\leqslant|x_1-x_2|A_{j+k+2}(\psi_0,\dot\psi_0),\end{aligned}$$

and our assumptions entail that $A_{j+k+2}(\psi_0, \dot{\psi}_0)$ is finite for all $j, k = 0, 1$. Hence we have shown

Lemma 26. *Suppose that $\mathcal{M}$ is such that $-\Delta_{\mathcal{M}} \geqslant 0$, and that $\psi(t)$ is given by (3.5) with Cauchy data $(\psi_0, \dot{\psi}_0)$ in $H^{4,3}_{\mathcal{M}}(\mathcal{G})$. Then both, the family of functions $\{\psi(t),\ t \in \mathbb{R}\}$ and the family of their derivatives $\{\psi(t)',\ t \in \mathbb{R}\}$, are uniformly bounded on $\mathcal{G}$, and uniformly equicontinuous on each edge of $\mathcal{G}$. The same is valid for the family $\{\partial_t \psi(t),\ t \in \mathbb{R}\}$ and its derivatives $\{(\partial_t \psi(t))',\ t \in \mathbb{R}\}$.*

On the other hand, consider $t_1, t_2 \in \mathbb{R}$, $j, k = 0, 1$. Then (A.2c) yields

$$\begin{aligned}\sup_{x \in I_i} \left|\partial_x^j \partial_t^k \psi_i(t_1, x) - \partial_x^j \partial_t^k \psi_i(t_2, x)\right| &\leqslant \left\|\left(\partial_t^k \psi\right)(t_1)^{(j)} - \left(\partial_t^k \psi\right)(t_2)^{(j)}\right\|_\infty \\ &\leqslant |t_1 - t_2| A_{j+k+2}(\psi_0, \dot{\psi}_0).\end{aligned}$$

Hence on every edge I_i and for all $x \in I_i$, the mappings $t \mapsto \psi_i(t, x)$, $\partial_x \psi_i(t, x)$, $\partial_t \psi_i(t, x)$, and $\partial_x \partial_t \psi_i(t, x)$ are uniformly continuous, uniformly in $x \in I_i$. Thus we have established: If the Cauchy data $(\psi_0, \dot{\psi}_0)$ belong to $H^{4,3}_{\mathcal{M}}(\mathcal{G})$, then for every edge I_i of $\mathcal{G}$ the maps

$$(t, x) \mapsto \begin{cases} \psi_i(t, x), \\ \frac{\partial}{\partial x} \psi_i(t, x), \\ \frac{\partial}{\partial t} \psi_i(t, x), \\ \frac{\partial^2}{\partial x \partial t} \psi_i(t, x), \end{cases} \qquad (t, x) \in \mathbb{R} \times I_i,$$

are uniformly continuous. So we can apply the lemma of Clairaut–Schwarz, see e.g. [19, Theorem 7.A.11, p. 194] for the version we use, to conclude

Lemma 27. *Suppose that $\mathcal{M}$ is such that $-\Delta_{\mathcal{M}} \geqslant 0$, and that $\psi(t)$ is given by (3.5) with Cauchy data $(\psi_0, \dot{\psi}_0)$ in $H^{4,3}_{\mathcal{M}}(\mathcal{G})$. Then for every edge I_i of $\mathcal{G}$ the mixed partial derivative (A.3) exists and equals the other mixed partial derivative (A.4), which is uniformly continuous for $(t, x) \in \mathbb{R} \times I_i$.*

Lemma 18 is a direct consequence of this lemma as is Lemma 23 in combination with the following observation. The volume

$$\mu\big(B(p, t_0 - t)\big) = \int\limits_{q \in B(p, t_0 - t)} dq$$

of $B(p, t_0 - t)$ is continuous in t. More precisely, the uniform estimate ($t_2 \leqslant t_1$)

$$\begin{aligned}0 \leqslant \mu\big(B(p, t_0 - t_2) \setminus B(p, t_0 - t_1)\big) &= \mu\big(B(p, t_0 - t_2)\big) - \mu\big(B(p, t_0 - t_1)\big) \\ &\leqslant (t_1 - t_2) 2\big(|\mathcal{E}| + |\mathcal{I}|\big)\end{aligned}$$

is easily established.

References

[1] F.M. Atay, A. Hutt, Stability and bifurcations in neural fields with finite propagation speed and general connectivity, SIAM J. Appl. Math. 65 (2005) 644–666.
[2] J. Cheeger, M. Gromov, M.E. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differential Geom. 17 (1982) 15–53.
[3] J. Cheeger, M.E. Taylor, On the diffraction of waves by conical singularities, Comm. Pure Appl. Math. 35 (1982) 275–331, 487–529.
[4] L.C. Evans, Partial Differential Equations, Amer. Math. Soc., Providence, RI, 1998.
[5] R. Haag, Local Quantum Physics, Springer, Berlin, Heidelberg, New York, 1992.
[6] C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw–Hill, New York, 1980.
[7] R. Jost, The General Theory of Quantized Fields, Lect. Appl. Math., vol. IV, Amer. Math. Soc., Providence, RI, 1965.
[8] V. Kostrykin, J. Potthoff, R. Schrader, Heat kernels on metric graphs and a trace formula, in: F. Germinet, P.D. Hislop (Eds.), Adventures in Mathematical Physics, in: Contemp. Math., vol. 447, Amer. Math. Soc., Providence, RI, 2006, pp. 175–198.
[9] V. Kostrykin, J. Potthoff, R. Schrader, Contraction semigrous on metric graphs, in: P. Exner, J. Keating, P. Kuchment, T. Sunada, A. Teplyaev (Eds.), Analysis on Graphs and Its Application, in: Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., 2008, pp. 423–458.
[10] V. Kostrykin, R. Schrader, Kirchhoff's rule for quantum wires, J. Phys. A: Math. Gen. 32 (1999) 595–630.
[11] V. Kostrykin, R. Schrader, The inverse scattering problem for metric graphs and the traveling salesman problem, arXiv:math-ph/0603010.
[12] V. Kostrykin, R. Schrader, Laplacians on metric graphs: Eigenvalues, resolvents and semigroups, in: G. Berkolaiko, R. Carlson, S.A. Fulling, P. Kuchment (Eds.), Quantum Graphs and Their Applications, in: Contemp. Math., vol. 415, Amer. Math. Soc., Providence, RI, 2006, pp. 201–225.
[13] E.H. Lieb, M. Loss, Analysis, Grad. Stud. Math., vol. 14, Amer. Math. Soc., Providence, 1997.
[14] R. Schrader, Finite propagation speed and causal free quantum fields on networks, J. Phys. A: Math. Gen. 42 (2009) 495401.
[15] S.S. Schweber, An Introduction to Relativistic Quantum Field Theory, Row, Petterson & Co, Evanston, 1961.
[16] R.F. Streater, A.S. Wightman, PCT, Spin Statistics and All That, Benjamin, New York, Amsterdam, 1964.
[17] M.E. Taylor, Pseudodifferential Operators, Princeton Univ. Press, Princeton, 1981.
[18] M.E. Taylor, Partial Differential Equations I, Springer, New York, Berlin, Heidelberg, 1996.
[19] W.L. Voxman, R.H. Goetschel, Advanced Calculus, Dekker, New York, Basel, 1981.
[20] S. Weinberg, The Quantum Theory of Fields I, Cambridge Univ. Press, Cambridge, 1995.
[21] A.S. Wightman, Quantum field theory in terms of vacuum expectation values, Phys. Rev. 101 (1956) 860–866.